Department of Mathematics

Mathematics Working Papers

Tennessee State University *Year* 2006

Operads in Iterated Monoidal Categories

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Abstract. The structure of a k-fold monoidal category as introduced by Balteanu, Fiedorowicz, Schwänzl and Vogt in [2] can be seen as a weaker structure than a symmetric or even braided monoidal category. In this paper we show that it is still sufficient to permit a good definition of (n-fold) operads in a k-fold monoidal category which generalizes the definition of operads in a braided category. Furthermore, the inheritance of structure by the category of operads is actually an inheritance of iterated monoidal structure, decremented by at least two iterations. We prove that the category of n-fold operads in a k-fold monoidal category is itself a $(k-n)$-fold monoidal, strict 2-category, and show that n-fold operads are automatically $(n-1)$-fold operads. We also introduce a family of simple examples of k-fold monoidal categories and classify operads in the example categories.

Contents

1. Introduction

In this introductory section we will give a brief, non-chronological overview of the relationship between operads, higher category theory, and topology. This will serve to motivate the study of iterated monoidal categories and their operads that comprises the remaining sections. In the second section, in order to be self contained, we repeat the definition of the iterated monoidal categories first set down in [2]. In the fourth section we seek to fill a gap in the literature which currently contains few good examples of that definition. Thus our first new contribution consists of a series of simple and very geometric iterated monoidal categories based on totally ordered monoids. By simple we mean that axioms are largely fulfilled due to relationships between max, plus, concatenation, sorting and lexicographic ordering as well as the fact that all diagrams commute since the underlying directed graph of the category is merely the total order. The most interesting examples of n-fold monoidal categories are those whose objects can be represented by Ferrer or Young diagrams (the underlying shapes of Young tableaux.) These exhibit products with the geometrical interpretation "combining stacks of boxes." Managers of warehouses or quarries perhaps may

Key words and phrases. enriched categories, n-categories, iterated monoidal categories.
Thanks to XY-pic for the diagrams.

already be well aquainted with the three dimensional version of the main example of iterated monoidal categories we introduce here. Imagine that floor space in the quarry or warehouse is at a premium and that therefore you are charged with combining several stacks of crates or stone blocks by restacking some together vertically and shifting others together horizontally. It turns out that the best result in terms of gained floor space is always to be achieved most efficiently by doing the restacking and shifting in a very particular order–horizontally first, then vertically.

The main new contribution is the theory of operads within, or enriched in, iterated monoidal categories. This theory is based upon the fact that the natural setting of operads turns out to be in a category with lax interchange between multiple operations, as opposed to the full strength of a braiding or symmetry as is classically assumed. Batanin's definition of n-operad also relies on this insight [4]. In that paper he notes that an iterated monoidal category $\mathcal{V}$ would be an example of a globular monoidal category with a single object, and a single arrow in each dimension up to n, in which last dimension the arrows would actually be the objects of $\mathcal{V}$. Of course the invertibility of the interchange would also have to be dropped from his definition. In that case the n-fold operads defined here would correspond to Batanin's n-operads. The advantages of seeing them in a single categorical dimension are in the way that doing so generalizes the fact that operads in a symmetric monoidal category inherit its symmetric structure. We investigate the somewhat flexible structure of the iterated monoidal 2-category that n-fold operads comprise. Flexibility arises from the difference between n and k, where one is investigating n-fold operads in a k-fold monoidal category $\mathcal{V}$, where $n < k - 1$. It turns out that choosing n much smaller than k allows multiple interchanging products to be defined on the category of operads, whereas choosing n closer to k allows the operads to take on multiple operad structures at once with respect to the products in $\mathcal{V}$. Examples of combinatorial operads living in the previously introduced combinatorically defined categories are utilized to demonstrate the sharpness of several of the resulting theorems, i.e. to provide counterexamples. The examples start to take on a life of their own, however, as theorems and open questions about the classification of operads in combinatorial n-fold monoidal categories arise. The definition of operad in the categories with morphisms given by ordering leads to descriptions of interesting kinds of growth. We give a complete description of the simple example of 2-fold operads in the natural numbers. We then give the elementary results for operads in the category of Young diagrams. In the basic examples linear and logarithmic growth characterize respective dimensions in a single sequence of Young diagrams. These phenomena hint towards a theory of operadic growth. Full investigation and further classification must await a future sequel to this paper. Applications might be found in scientific fields such as the theory of small world networks, where the diameter of a network is the logarithm of the number of nodes.

First, however, we look at some of the history and philosophy of the two major players here, operads and iterated monoidal categories. Operads in a category of topological spaces are the crystallization of several approaches to the recognition problem for iterated loop spaces. Beginning with Stasheff's associahedra and Boardman and Vogt's little n-cubes, and continuing with more general A_∞, E_n and E_∞ operads described by May and others, that problem has largely been solved [26], [8], [21]. Loop spaces are characterized by admitting an operad action of the appropriate kind. More lately Batanin's approach to higher categories through internal and higher operads promises to shed further light on the combinatorics of E_n spaces [5], [6].

Recently there has also been growing interest in the application of higher dimensional structured categories to the characterization of loop spaces. The program being advanced by many categorical homotopy theorists seeks to model the coherence laws governing homotopy types with the coherence axioms of structured n-categories. By modeling we mean a connection that will be in the form of a functorial equivalence between categories of special categories and categories of special spaces. The largest challenges currently are to find the most natural and efficient definition of (weak) n-category, and to determine the nature of the functor from categories to spaces. The latter will almost certainly be analogous to the nerve functor on 1-categories, which preserves homotopy equivalence. In [27] Street defines the nerve of a strict n-category. Recently Duskin in [9] has worked out the description of the nerve of a bicategory. A second part of the latter paper promises the full description of the functor including how it takes morphisms of bicategories to continuous maps.

One major recent advance is the discovery of Balteanu, Fiedorowicz, Schwänzl and Vogt in [2] that the nerve functor on categories gives a direct connection between iterated monoidal categories and iterated loop spaces. Stasheff [26] and Mac Lane [19] showed that monoidal categories are precisely analogous to 1-fold loop spaces. There is a similar connection between symmetric monoidal categories and infinite loop spaces. The first step in filling in the gap between 1 and infinity was made in [10] where it is shown that the group completion of the nerve of a braided monoidal category is a 2-fold loop space. In [2] the authors finish this process by, in their words, "pursuing an analogy to the tautology that an n-fold loop space is a loop space in the category of $(n-1)$-fold loop spaces." The first thing they focus on is the fact that a braided category is a special case of a carefully defined 2-fold monoidal category. Based on their observation of the correspondence between loop spaces and monoidal categories, they iteratively define the notion of n-fold monoidal category as a monoid in the category of $(n-1)$-fold monoidal categories. In [2] a symmetric category is seen as a category that is n-fold monoidal for all n. The main result in that paper is that the group completion of the nerve of an n-fold monoidal category is an n-fold loop space. It is still open whether this is a complete characterization, that is, whether every n-fold loop space arises as the nerve of an n-fold monoidal category. Much progress towards the answer to this question was made by the original authors in their sequel paper, but the desired result was later shown to remain unproven. One of the future goals of the program begun here is to use weakenings or deformations of the examples of n-fold monoidal categories introduced here to model specific loop spaces in a direct way.

The connection between the n-fold monoidal categories of Fiedorowicz and the theory of higher categories is through the periodic table as laid out in [1]. Here Baez organizes the k-tuply monoidal n-categories, by which terminology he refers to $(n+k)$-categories that are trivial below dimension k. The triviality of lower cells allows the higher ones to compose freely, and thus these special cases of $(n+k)$-categories are viewed as n-categories with k multiplications. Of course a k-tuply monoidal n-category is a special k-fold monoidal n-category. The specialization results from the definition(s) of n-category, all of which seem to include the axiom that the interchange transformation between two ways of composing four higher morphisms along two different lower dimensions is required to be an isomorphism. As will be mentioned in the next section the property of having iterated loop space nerves held by the

k-fold monoidal categories relies on interchange transformations that are not isomorphisms. If those transformations are indeed isomorphisms then the k-fold monoidal 1-categories do reduce to the braided and symmetric 1-categories of the periodic table. Whether this continues for higher dimensions, yielding for example the sylleptic monoidal 2-categories of the periodic table as 3-fold monoidal 2-categories with interchange isomorphisms, is yet to be determined.

A further refinement of higher categories is to require all morphisms to have inverses. These special cases are referred to as n-groupoids, and since their nerves are simpler to describe it has long been suggested that they model homotopy n-types through a construction of a fundamental n-groupoid. This has in fact been shown to hold in Tamsamani's definition of weak n-category [28], and in a recent paper by Cisinski to hold in the definition of Batanin as found in [4]. A homotopy n-type is a topological space X for which $\pi_k(X)$ is trivial for all $k > n$. Thus the homotopy n-types are classified by π_k for $k \leq n$. It has been suggested that a key requirement for any useful definition of n-category is that a k-tuply monoidal n-groupoid be associated functorially (by a nerve) to a topological space which is a homotopy n-type and a k-fold loop space [1]. The loop degree will be precise for $k < n+1$, but for $k > n$ the associated homotopy n-type will be an infinite loop space. This last statement is a consequence of the stabilization hypothesis , which states that there should be a left adjoint to forgetting monoidal structure that is an equivalence of $(n+k+2)$-categories between k-tuply monoidal n-categories and $(k+1)$-tuply monoidal n-categories for $k > n+1$. This hypothesis has been shown by Simpson to hold in the case of Tamsamani's definition [24]. For the case of $n = 1$ if the interchange transformations are isomorphic then a k-fold monoidal 1-category is equivalent to a symmetric category for $k > 2$. With these facts in mind it is possible that if we wish to precisely model homotopy n-type k-fold loop spaces for $k > n$ then we may need to consider k-fold as well as k-tuply monoidal n-categories. This paper is part of an embryonic effort in that direction.

Since a loop space can be efficiently described as an operad algebra, it is not surprising that there are several existing definitions of n-category that utilize operad actions. These definitions fall into two main classes: those that define an n-category as an algebra of a higher order operad, and those that achieve an inductive definition using classical operads in symmetric monoidal categories to parameterize iterated enrichment. The first class of definitions is typified by Batanin and Leinster [4],[17].

The former author defines monoidal globular categories in which interchange transformations are isomorphisms and which thus resemble free strict n-categories. Globular operads live in these, and take all sorts of pasting diagrams as input types, as opposed to just a string of objects as in the case of classical operads. The binary composition in an n-category derives from the action of a certain one of these globular operads. Leinster expands this concept to describe n-categories with unbiased composition of any number of cells. The second class of definitions is typified by the works of Trimble and May [29], [22].

The former parameterizes iterated enrichment with a series of operads in $(n-1)$-Cat achieved by taking the fundamental $(n-1)$-groupoid of the k^{th} component of the topological path composition operad E. The latter begins with an A_∞ operad in a symmetric monoidal category $\mathcal{V}$ and requires his enriched categories to be tensored over $\mathcal{V}$ so that the iterated enrichment always refers to the same original operad.

Iterated enrichment over n-fold categories is described in [11] and [12]. It seems wothwhile to define n-fold operads in n-fold monoidal categories in a way that is consistent with the spirit of Batanin's globular operads. Their potential value may include using them to weaken enrichment over n-fold monoidal categories in a way that is in the spirit of May and Trimble. This program carries with it the promise of characterizing k-fold loop spaces with homotopy n-type for all n, k by describing the categories with exactly those spaces as nerves. As a candidate for the type of category with such a nerve we suggest a weak n-category with k multiplications that interchange only in the lax sense.

In this paper we follow May by defining n-fold operads in terms of monoids in a certain category of collections. A more abstract approach for future consideration would begin by finding an equivalent definition in the language of Weber, where an operad lives within a monoidal pseudo algebra of a 2-monad [30]. This latter is a general notion of operad which includes as instances both the classical operads and the higher operads of Batanin.

2. k-fold monoidal categories

This sort of category was developed and defined in [2]. The authors describe its structure as arising recursively from its description as a monoid in the category of $(k-1)$-fold monoidal categories. Here we present that definition (in its expanded form) altered only slightly to make visible the coherent associators as in [11]. That latter paper describes its structure in terms of tensor objects in the category of $(k-1)$-fold monoidal categories. Our variation has the effect of making visible the associators α^i_{ABC}. It is desirable to do so for several reasons. One is that this makes easier a direct comparison with Batanin's definition of monoidal globular categories as in [4]. A monoidal globular category can be seen as a quite special case of a iterated monoidal category, with source and target maps that take objects to those in a category with one less product, and with interchanges that are isomorphisms.

The other reason is that in this paper we will consider a category of collections in an iterated monoidal category which will be (iterated) monoidal only up to natural associators. That being said, in much of the remainder of this paper we will consider examples with strict associativity, where each α is the identity, and in interest of clarity will often hide associators. One actual simplification in the following definition is that all the products are assumed to have the same unit. We note that this is easily generalized, as in the case of collections which we will consider.

2.1. **Definition.** An *n-fold monoidal category* is a category $\mathcal{V}$ with the following structure.

(1) There are n multiplications

$$\otimes_1, \otimes_2, \ldots, \otimes_n : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$$

each equipped with an associator α_{UVW}, a natural isomorphism which satisfies the pentagon equation:

$$\begin{tikzcd}
& ((U \otimes_i V) \otimes_i W) \otimes_i X \arrow[rr, "\alpha^i_{UVW} \otimes 1_X"] \arrow[dl, "\alpha^i_{(U \otimes_i V)WX}"] & & (U \otimes_i (V \otimes_i W)) \otimes_i X \arrow[dr, "\alpha^i_{U(V \otimes_i W)X}"] & \\
(U \otimes_i V) \otimes_i (W \otimes_i X) \arrow[drr, "\alpha^i_{UV(W \otimes_i X)}"] & & & & U \otimes_i ((V \otimes_i W) \otimes_i X) \arrow[dll, "1_U \otimes_i \alpha^i_{VWX}"] \\
& & U \otimes_i (V \otimes_i (W \otimes_i X)) & &
\end{tikzcd}$$

(2) $\mathcal{V}$ has an object I which is a strict unit for all the multiplications.
(3) For each pair (i, j) such that $1 \le i < j \le n$ there is a natural transformation

$$\eta^{ij}_{ABCD} : (A \otimes_j B) \otimes_i (C \otimes_j D) \to (A \otimes_i C) \otimes_j (B \otimes_i D).$$

These natural transformations η^{ij} are subject to the following conditions:

(a) Internal unit condition: $\eta^{ij}_{ABII} = \eta^{ij}_{IIAB} = 1_{A \otimes_j B}$
(b) External unit condition: $\eta^{ij}_{AIBI} = \eta^{ij}_{IAIB} = 1_{A \otimes_i B}$
(c) Internal associativity condition: The following diagram commutes.

$$\begin{tikzcd}
((U \otimes_j V) \otimes_i (W \otimes_j X)) \otimes_i (Y \otimes_j Z) \arrow[r, "\eta^{ij}_{UVWX} \otimes_i 1_{Y \otimes_j Z}"] \arrow[d, "\alpha^i"] & \big((U \otimes_i W) \otimes_j (V \otimes_i X)\big) \otimes_i (Y \otimes_j Z) \arrow[d, "\eta^{ij}_{(U \otimes_i W)(V \otimes_i X)YZ}"] \\
(U \otimes_j V) \otimes_i ((W \otimes_j X) \otimes_i (Y \otimes_j Z)) \arrow[d, "1_{U \otimes_j V} \otimes_i \eta^{ij}_{WXYZ}"] & ((U \otimes_i W) \otimes_i Y) \otimes_j ((V \otimes_i X) \otimes_i Z) \arrow[d, "\alpha^i \otimes_j \alpha^i"] \\
(U \otimes_j V) \otimes_i \big((W \otimes_i Y) \otimes_j (X \otimes_i Z)\big) \arrow[r, "\eta^{ij}_{UV(W \otimes_i Y)(X \otimes_i Z)}"] & (U \otimes_i (W \otimes_i Y)) \otimes_j (V \otimes_i (X \otimes_i Z))
\end{tikzcd}$$

(d) External associativity condition: The following diagram commutes.

$$\begin{tikzcd}
((U \otimes_j V) \otimes_j W) \otimes_i ((X \otimes_j Y) \otimes_j Z) \arrow[r, "\eta^{ij}_{(U \otimes_j V)W(X \otimes_j Y)Z}"] \arrow[d, "\alpha^j \otimes_i \alpha^j"] & \big((U \otimes_j V) \otimes_i (X \otimes_j Y)\big) \otimes_j (W \otimes_i Z) \arrow[d, "\eta^{ij}_{UVXY} \otimes_j 1_{W \otimes_i Z}"] \\
(U \otimes_j (V \otimes_j W)) \otimes_i (X \otimes_j (Y \otimes_j Z)) \arrow[d, "\eta^{ij}_{U(V \otimes_j W)X(Y \otimes_j Z)}"] & ((U \otimes_i X) \otimes_j (V \otimes_i Y)) \otimes_j (W \otimes_i Z) \arrow[d, "\alpha^j"] \\
(U \otimes_i X) \otimes_j \big((V \otimes_j W) \otimes_i (Y \otimes_j Z)\big) \arrow[r, "1_{U \otimes_i X} \otimes_j \eta^{ij}_{VWYZ}"] & (U \otimes_i X) \otimes_j ((V \otimes_i Y) \otimes_j (W \otimes_i Z))
\end{tikzcd}$$

(e) Finally it is required for each triple (i, j, k) satisfying $1 \le i < j < k \le n$ that the giant hexagonal interchange diagram commutes.

$$\begin{tikzcd}
 & ((A \otimes_k A') \otimes_j (B \otimes_k B')) \otimes_i ((C \otimes_k C') \otimes_j (D \otimes_k D')) \arrow[dl, "\eta^{jk}_{AA'BB'} \otimes_i \eta^{jk}_{CC'DD'}"'] \arrow[dr, "\eta^{ij}_{(A \otimes_k A')(B \otimes_k B')(C \otimes_k C')(D \otimes_k D')}"] & \\
((A \otimes_j B) \otimes_k (A' \otimes_j B')) \otimes_i ((C \otimes_j D) \otimes_k (C' \otimes_j D')) \arrow[d, "\eta^{ik}_{(A \otimes_j B)(A' \otimes_j B')(C \otimes_j D)(C' \otimes_j D')}"] & & ((A \otimes_k A') \otimes_i (C \otimes_k C')) \otimes_j ((B \otimes_k B') \otimes_i (D \otimes_k D')) \arrow[d, "\eta^{ik}_{AA'CC'} \otimes_j \eta^{ik}_{BB'DD'}"] \\
((A \otimes_j B) \otimes_i (C \otimes_j D)) \otimes_k ((A' \otimes_j B') \otimes_i (C' \otimes_j D')) \arrow[dr, "\eta^{ij}_{ABCD} \otimes_k \eta^{ij}_{A'B'C'D'}"'] & & ((A \otimes_i C) \otimes_k (A' \otimes_i C')) \otimes_j ((B \otimes_i D) \otimes_k (B' \otimes_i D')) \arrow[dl, "\eta^{jk}_{(A \otimes_i C)(A' \otimes_i C')(B \otimes_i D)(B' \otimes_i D')}"] \\
 & ((A \otimes_i C) \otimes_j (B \otimes_i D)) \otimes_k ((A' \otimes_i C') \otimes_j (B' \otimes_i D')) &
\end{tikzcd}$$

Note that for $q > p$ we have natural transformations

$$\eta^{pq}_{AIIB} : A \otimes_p B \to A \otimes_q B \qquad \text{and} \qquad \eta^{pq}_{IABI} : A \otimes_p B \to B \otimes_q A.$$

If the authors of [2] had insisted a 2-fold monoidal category be a tensor object in the category of monoidal categories and *strictly monoidal* functors, this would be equivalent to requiring that $\eta = 1$. In view of the above, they note that this would imply $A \otimes_1 B = A \otimes_2 B = B \otimes_1 A$ and similarly for morphisms.

Joyal and Street [14] considered a similar concept to Balteanu, Fiedorowicz, Schwänzl and Vogt's idea of 2–fold monoidal category. The former pair required the natural transformation η_{ABCD} to be an isomorphism and showed that the resulting category is naturally equivalent to a braided monoidal category. As explained in [2], given such a category one obtains an equivalent braided monoidal category by discarding one of the two operations, say $\otimes_2$, and defining the commutativity isomorphism for the remaining operation $\otimes_1$ to be the composite

$$A \otimes_1 B \xrightarrow{\eta_{IABI}} B \otimes_2 A \xrightarrow{\eta^{-1}_{BIIA}} B \otimes_1 A.$$

The authors of [2] remark that a symmetric monoidal category is n-fold monoidal for all n. This they demonstrate by letting

$$\otimes_1 = \otimes_2 = \cdots = \otimes_n = \otimes$$

and defining

$$\eta^{ij}_{ABCD} = \alpha^{-1} \circ (1_A \otimes \alpha) \circ (1_A \otimes (c_{BC} \otimes 1_D)) \circ \circ (1_A \otimes \alpha^{-1}) \circ \alpha$$

for all $i < j$. Here $c_{BC} : B \otimes C \to C \otimes B$ is the symmetry natural transformation.

Joyal and Street [14] require that the interchange natural transformations η^{ij}_{ABCD} be isomorphisms and observed that for $n \geq 3$ the resulting sort of category is equivalent to a symmetric monoidal category. Thus as Balteanu, Fiedorowicz, Schwänzl and Vogt point out, the nerves of such categories have group completions which are infinite loop spaces rather than only n–fold loop spaces.

Because of the recursive nature of the definition of iterated monoidal category, there are multiple forgetful functors implied. Specifically, letting $n < k$, from the category of k-fold monoidal categories to the category of n-fold monoidal categories there are $\binom{k}{n}$ forgetful functors which forget all but the chosen set of products.

The coherence theorem for iterated monoidal categories states that any diagram composed solely of interchange transformations commutes; i.e. if two compositions of various interchange transformations (legs of a diagram) have the same source and target then they describe the same morphism. Furthermore we can easily determine when a composition of interchanges exists between objects. Here are the necessary definitions and Theorem as given in [2].

2.2. **Definition.** Let $\mathcal{F}_n(S)$ be the free n-fold monoidal category on the finite set S. Its objects are all finite expressions generated by the elements of S using the products $\otimes_i, i = 1..n$. By $\mathcal{M}_n(S)$ we denote the sub-category of $\mathcal{F}_n(S)$ whose objects are expressions in which each element of S occurs exactly once.

If $S \subset T$ then there is a restriction functor $\mathcal{M}_n(T) \to \mathcal{M}_n(S)$, induced by the functor $\mathcal{F}_n(T) \to \mathcal{F}_n(S)$, which sends $T - S$ to the empty expression 0.

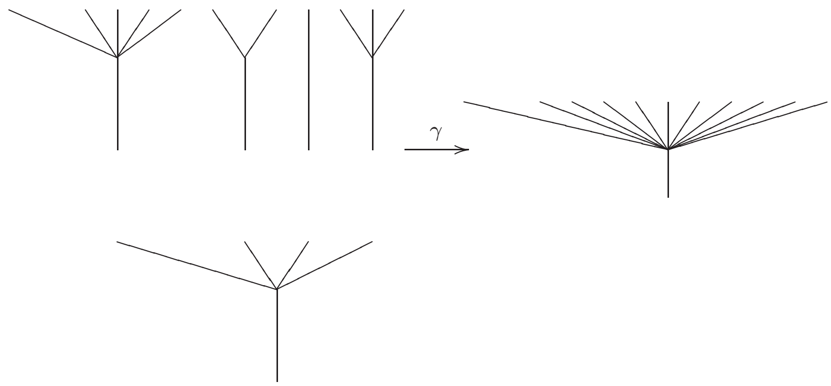
2.3. **Definition.** Let A be an object of $\mathcal{M}_n(S)$. For $a, b \in S$ we say that $a \otimes_i b$ *in* A if the restriction functor $\mathcal{M}_n(S) \to \mathcal{M}_n(a, b)$ sends A to $a \otimes_i b$.

2.4. **Theorem.** [2] *Let A and B be objects of $\mathcal{M}_n(S)$. Then*

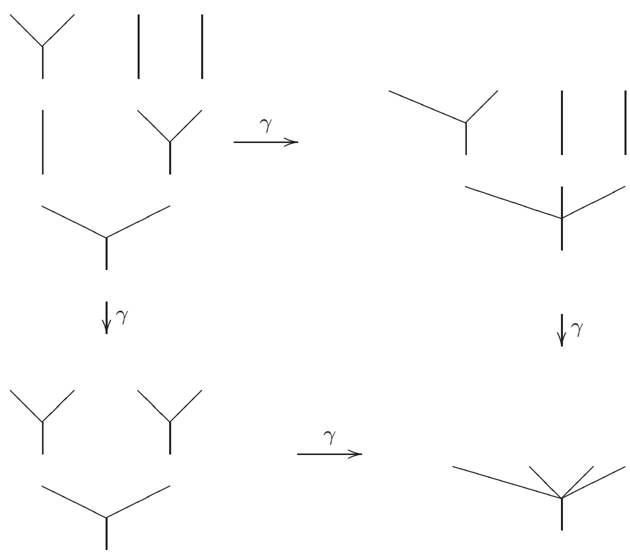
(1) *There is at most one morphism $A \to B$.*
(2) *Moreover, there exists a morphism $A \to B$ if and only if, for every $a, b \in S$, $a \otimes_i b \in A$ implies that either $a \otimes_j b$ or $b \otimes_j a$ is in B for some $j > i$.*

3. n-fold operads

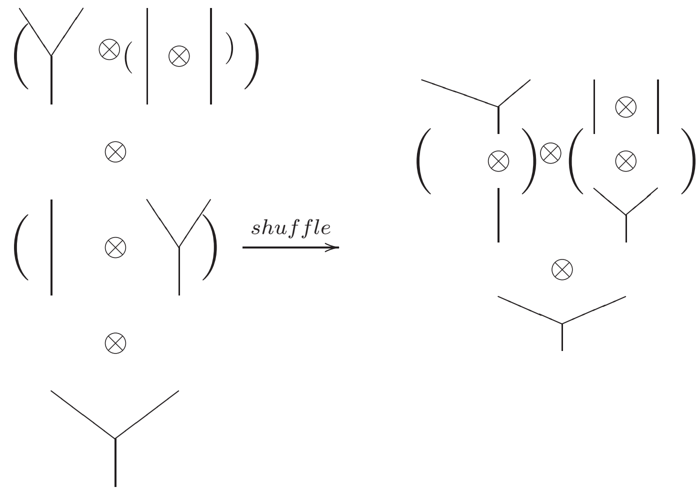
The two principle components of an operad are a collection, historically a sequence, of objects in a monoidal category and a family of composition maps. Operads are often described as parameterizations of n-ary operations. Peter May's original definition of operad in a symmetric (or braided) monoidal category [21] has a composition γ that takes the tensor product of the n^{th} object (n-ary operation) and n others (of various arity) to a resultant that sums the arities of those others. The n^{th} object or n-ary operation is often pictured as a tree with n leaves, and the composition appears like this:



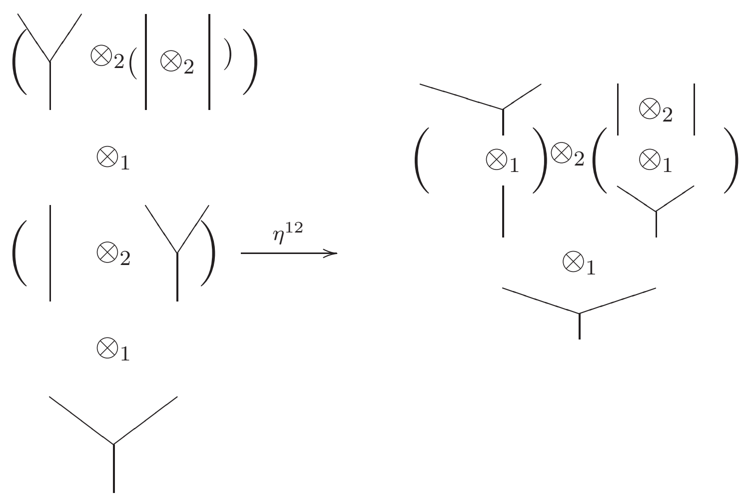
By requiring this composition to be associative we mean that it obeys this sort of pictured commuting diagram:



In the above pictures the tensor products are shown just by juxtaposition, but now we would like to think about the products more explicitly. If the monoidal category is not strict, then there is actually required another leg of the associativity diagram, where the tensoring is reconfigured so that the composition can operate in an alternate order. Here is how that rearranging looks in a symmetric (braided) category, where the shuffling is accomplished by use of the symmetry (braiding):



We now foreshadow our definition of operads in an iterated monoidal category with the same picture as above but using two tensor products, $\otimes_1$ and $\otimes_2$. It becomes clear that the true nature of the shuffle is in fact that of an interchange transformation.



To see this just focus on the actual domain and range of η^{12} which are the upper two levels of trees in the pictures, with the tensor product $(| \otimes_2 |)$ considered as a single object.

Now we are ready to give the technical definitions. We begin with the definition of 2-fold operad in an n-fold monoidal category, as in the above picture, and then mention how it generalizes the case of operad in a braided category. Because of this generalization of the well known case, and since there are easily described examples of 2-fold monoidal categories based on a braided category as in [13], it seems worthwhile to work out the theory for the 2-fold operads in its entirety before moving on to n-fold operads.

Let $\mathcal{V}$ be an n-fold monoidal category as defined in Section 2.

3.1. **Definition.** A 2-fold operad $\mathcal{C}$ in $\mathcal{V}$ consists of objects $\mathcal{C}(j)$, $j \geq 0$, a unit map $\mathcal{J} : I \to \mathcal{C}(1)$, and composition maps in $\mathcal{V}$

$$\gamma^{12} : \mathcal{C}(k) \otimes_1 (\mathcal{C}(j_1) \otimes_2 \cdots \otimes_2 \mathcal{C}(j_k)) \to \mathcal{C}(j)$$

for $k \geq 1$, $j_s \geq 0$ for $s = 1 \dots k$ and $\sum_{n=1}^{k} j_n = j$. The composition maps obey the following axioms:

(1) Associativity: The following diagram is required to commute for all $k \geq 1$, $j_s \geq 0$ and $i_t \geq 0$, and where $\sum_{s=1}^{k} j_s = j$ and $\sum_{t=1}^{j} i_t = i$. Let $g_s = \sum_{u=1}^{s} j_u$ and let $h_s = \sum_{u=1+g_{s-1}}^{g_s} i_u$. The η^{12} labelling the leftmost arrow actually stands for a variety of equivalent maps which factor into instances of the 12 interchange.

$$\begin{tikzcd}
\mathcal{C}(k) \otimes_1 \left(\bigotimes_{s=1}^{k} {}_2\mathcal{C}(j_s) \right) \otimes_1 \left(\bigotimes_{t=1}^{j} {}_2\mathcal{C}(i_t) \right) \arrow[r, "\gamma^{12} \otimes_1 \mathrm{id}"] \arrow[dd, "\mathrm{id} \otimes_1 \eta^{12}"'] & \mathcal{C}(j) \otimes_1 \left(\bigotimes_{t=1}^{j} {}_2\mathcal{C}(i_t) \right) \arrow[d, "\gamma^{12}"] \\
 & \mathcal{C}(i) \\
\mathcal{C}(k) \otimes_1 \left(\bigotimes_{s=1}^{k} {}_2\mathcal{C}(j_s) \otimes_1 \left(\bigotimes_{u=1}^{j_s} {}_2\mathcal{C}(i_{u+g_{s-1}}) \right) \right) \arrow[r, "\mathrm{id} \otimes_1 (\otimes_2^k \gamma^{12})"'] & \mathcal{C}(k) \otimes_1 \left(\bigotimes_{s=1}^{k} {}_2\mathcal{C}(h_s) \right) \arrow[u, "\gamma^{12}"']
\end{tikzcd}$$

(2) Respect of units is required just as in the symmetric case. The following unit diagrams commute.

$$\begin{tikzcd}
\mathcal{C}(k) \otimes_1 (\otimes_2^k I) \arrow[r, equal] \arrow[d, "1 \otimes_1 (\otimes_2^k \mathcal{J})"'] & \mathcal{C}(k) \\
\mathcal{C}(k) \otimes_1 (\otimes_2^k \mathcal{C}(1)) \arrow[ur, "\gamma^{12}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes_1 \mathcal{C}(k) \arrow[r, equal] \arrow[d, "\mathcal{J} \otimes_1 1"'] & \mathcal{C}(k) \\
\mathcal{C}(1) \otimes_1 \mathcal{C}(k) \arrow[ur, "\gamma^{12}"] &
\end{tikzcd}$$

Note that operads in a braided monoidal category are examples of 2-fold operads. This is true based on the arguments of Joyal and Street [14], who showed that braided categories arise as 2-fold monoidal categories where the interchanges are isomorphisms. Also note that given such a perspective on a braided category, the two products are equivalent and the use of the braiding to shuffle in the operad associativity requirement can be rewritten as the use of the interchange.

It is immediately clear that we can define operads using more than just the first two products in an n-fold monoidal category. The best way of going about this is to use the theory of monoids, (and more generally enriched categories), in iterated monoidal categories. We continue by first describing this procedure for 2-fold operads. Operads in a symmetric (braided) monoidal category are often efficiently defined as the monoids of a category of collections. For a braided category $(\mathcal{V}, \otimes)$ with coproducts that are preserved by both functors $(— \otimes A)$ and $(A \otimes —)$ the objects of $\mathrm{Col}(\mathcal{V})$ are functors from the category of natural numbers to $\mathcal{V}$. In other words the data for a collection $\mathcal{C}$ is a sequence of objects $\mathcal{C}(j)$. Morphisms in $\mathrm{Col}(\mathcal{V})$ are natural transformations. The tensor product in $\mathrm{Col}(\mathcal{V})$ is given by

$$(\mathcal{B} \otimes \mathcal{C})(j) = \coprod_{\substack{k \geq 0 \\ j_1 + \cdots + j_k = j}} \mathcal{B}(k) \otimes (\mathcal{C}(j_1) \otimes \cdots \otimes \mathcal{C}(j_k))$$

where $j_i \geq 0$. This product is associative by use of the symmetry or braiding, and due to the hypothesis that the tensor product preserves the coproduct. The unit is the collection $(\emptyset, I, \emptyset, \dots)$ where $\emptyset$ is an initial object in $\mathcal{V}$.

Now recall how the interchange transformations generalize braiding. For $\mathcal{V}$ a 2-fold monoidal category with all coproducts in which both $\otimes_1$ and $\otimes_2$ preserve the coproduct, define the objects and morphisms of $\mathrm{Col}_2(\mathcal{V})$ in precisely the same way as in the braided case, but define the product to be

$$(\mathcal{B}\otimes^{12}\mathcal{C})(j) = \coprod_{\substack{k\geq 0\\ j_1+\cdots+j_k=j}} \mathcal{B}(k) \otimes_1 (\mathcal{C}(j_1) \otimes_2 \cdots \otimes_2 \mathcal{C}(j_k))$$

In general the interchangers will not be isomorphisms, so this product can not be that of a monoidal category with the usual strong associativity. However the interchangers can be used to make the product in question obey lax associativity, where the associator is a coherent natural transformation. This lax associativity is seen by inspection of the two 3-fold products $(\mathcal{B}\otimes^{12}\mathcal{C})\otimes^{12}\mathcal{D}$ and $\mathcal{B}\otimes^{12}(\mathcal{C}\otimes^{12}\mathcal{D})$. In the braided case mentioned above, the two large coproducts in question are seen to be composed of the same terms up to a braiding between them. Here the terms of the two coproducts are related by instances of the interchange transformation η^{12} from the term in $((\mathcal{B}\otimes^{12}\mathcal{C})\otimes^{12}\mathcal{D})(j)$ to the corresponding term in $(\mathcal{B}\otimes^{12}(\mathcal{C}\otimes^{12}\mathcal{D}))(j)$. For example upon expansion of the two three-fold products we see that in the coproduct which is $((\mathcal{B}\otimes^{12}\mathcal{C})\otimes^{12}\mathcal{D})(2)$ we have the term

$$\mathcal{B}(2) \otimes_1 (\mathcal{C}(1) \otimes_2 \mathcal{C}(1)) \otimes_1 (\mathcal{D}(1) \otimes_2 \mathcal{D}(1))$$

while in $(\mathcal{B}\otimes^{12}(\mathcal{C}\otimes^{12}\mathcal{D}))(2)$ we have the term

$$\mathcal{B}(2) \otimes_1 (\mathcal{C}(1) \otimes_1 \mathcal{D}(1)) \otimes_2 (\mathcal{C}(1) \otimes_1 \mathcal{D}(1)).$$

Note that the first of these terms appears courtesy of the fact that when a tensor product preserves coproducts, there is implied a distributive law $(\coprod B_n) \otimes A \cong \coprod (B_n \otimes A)$ as shown in [20].

The coherence theorem of iterated monoidal categories as it is stated in [2] guarantees the commutativity of the pentagon equation for the associators, since they are defined as compositions of interchangers η^{12} in $\mathcal{V}$. Some remarks about the non-invertibility of α are in order. Note that Mac Lane proves his coherence theorem in two steps [20]. First it is shown that every diagram involving only α (no α^{-1}) commutes. Then it is noted that this suffices to make every diagram of both α and α^{-1} commute since for every binary word there exists a path of just instances of α from that word to the word parenthesized all to the right. (Here we are taking the domain of α to be $(A \otimes B) \otimes C$.) Thus when α is not invertible we still have that every diagram commutes. There are still canonical maps from every binary word to the word parenthesized all to the right. However there are necessarily fewer diagrams. For instance if $(\mathcal{V}, \otimes)$ is lax monoidal there is no canonical map between the two objects $(B \otimes B) \otimes (B \otimes B)$ and $(B \otimes (B \otimes B)) \otimes B$. This affects the statement of the general associativity theorem for monoids in a lax monoidal category. Only the specific case of the general associativity theorem as stated by Mac Lane holds, as follows.

3.2. **Theorem.** *Let (A, μ) be a monoid in a (lax) monoidal category. Let A^n be the product given by $A^2 = A \otimes A$, $A^{n+1} = A \otimes A^n$, i.e. parenthesized to the right. Define the composition*

$\mu^{(n)}$ by $\mu^{(2)} = \mu, \mu^{(n+1)} = \mu \circ (1 \otimes \mu^{(n)})$. Then

$$\mu^{(n)} \circ (\mu^{(k_1)} \otimes \cdots \otimes \mu^{(k_n)}) = \mu^{(k_1+\cdots+k_n)} \circ \alpha'$$

for all $n, k_i \geq 2$ where α' stands for the canonical map to $A^{k_1+\cdots+k_n}$.

Proof. This is just the special case of the general associative law for monoids shown by Mac Lane, which only depends on the existence of the canonical map α' [20]. □

Now we have a condensed way of defining 2-fold operads.

3.3. **Theorem.** *2-fold operads in 2-fold monoidal $\mathcal{V}$ are monoids in $\mathrm{Col}_2(\mathcal{V})$.*

Proof. A monoid in $\mathrm{Col}_2(\mathcal{V})$ is an object $\mathcal{C}$ in $\mathrm{Col}_2(\mathcal{V})$ with multiplication and unit morphisms. Since morphisms of $\mathrm{Col}_2(\mathcal{V})$ are natural transformations the multiplication and unit consist of families of maps in $\mathcal{V}$ indexed by the natural numbers, with source and target exactly as required for operad composition and unit. The operad axioms are equivalent to the associativity and unit requirements of monoids. □

This brings us back to the question of defining operads in n-fold monoidal $\mathcal{V}$ using the higher products and interchanges. This idea will correspond to a series of higher products, denoted by $\otimes^{pq}$, in the category of collections. These are defined just as for the first case $\otimes^{12}$ above. Associators are as described above for the first product, using η^{pq} for the associator $\alpha\colon \mathcal{A} \otimes^{pq} (\mathcal{B} \otimes^{pq} \mathcal{C}) \to (\mathcal{A} \otimes^{pq} \mathcal{B}) \otimes^{pq} \mathcal{C}$. The unit for each is the collection $(\emptyset, I, \emptyset, \dots)$ where $\emptyset$ is an initial object in $\mathcal{V}$. Notice that these products do not interchange; i.e they are not functorial with respect to each other. Notice also that the associators in these categories of collections are not isomorphisms unless we are considering the special cases of braiding or symmetry. Instead they are lax monoidal, by which we will mean that the associator is merely a natural transformation which obeys the pentagon coherence condition.

Now we will focus on the products $\otimes^{(m-1)m}$ in the category of collections in n-fold monoidal $\mathcal{V}$, for $m \leq n$, since these will be seen to suffice for defining all operad compositions. Before defining m-fold operads as monoids with respect to $\otimes^{(m-1)m}$, we note that there is also fibrewise monoidal structure. This will be important in the description of the monoidal structure of the category of operads. In fact, we have the following

3.4. **Theorem.** *If an n-fold monoidal category $\mathcal{V}$ has coproducts and $(\mathcal{V}, \coprod, \otimes_3, \dots, \otimes_n)$ is an $(n-1)$-fold monoidal category for which each of the functors $(\text{—} \otimes_i A)$ and $(A \otimes_i \text{—})$ preserves coproducts, then for $n \geq m \geq 2$, the category of collections in $\mathcal{V}$ can be given the structure of an $(n-m+1)$-fold lax monoidal category, denoted $\mathrm{Col}_m(\mathcal{V})$.*

Proof. The first tensor product is $\hat{\otimes}_1 = \otimes^{(m-1)m}$ and the others are the higher fibrewise products starting with fibrewise $\otimes_{m+1}$. Thus the products of $\mathrm{Col}_m(\mathcal{V})$ are as follows:

$$(\mathcal{B}\hat{\otimes}_1\mathcal{C})(j) = \coprod_{\substack{k\geq 0\\ j_1+\cdots+j_k=j}} \mathcal{B}(k) \otimes_{m-1} (\mathcal{C}(j_1) \otimes_m \cdots \otimes_m \mathcal{C}(j_k))$$

and

$$(\mathcal{B}\hat{\otimes}_2\mathcal{C})(j) = \mathcal{B}(j)\otimes_{m+1}\mathcal{C}(j)$$
$$\vdots$$
$$(\mathcal{B}\hat{\otimes}_{n-m+1}\mathcal{C})(j) = \mathcal{B}(j)\otimes_{n}\mathcal{C}(j)$$

The unit for $\hat{\otimes}_1$ is $\mathcal{I} = (\emptyset, I, \emptyset, \dots)$ and the unit for all the other products is $\mathcal{M} = (I, I, \dots)$. First we must check that there are natural transformations

$$\xi^{1j} : (A\hat{\otimes}_j B)\hat{\otimes}_1(C\hat{\otimes}_j D) \to (A\hat{\otimes}_1 C)\hat{\otimes}_j(B\hat{\otimes}_1 D)$$

These utilize the η^{ij} of $\mathcal{V}$ and thus exist by inspection of the terms of the compound products. For example, in

$$((A\hat{\otimes}_2 B)\hat{\otimes}_1(C\hat{\otimes}_2 D))(2)$$

we find the term

$$(A(2) \otimes_3 B(2)) \otimes_1 ((C(1) \otimes_3 D(1)) \otimes_2 (C(1) \otimes_3 D(1))),$$

while in

$$(A\hat{\otimes}_1 C)\hat{\otimes}_2(B\hat{\otimes}_1 D)$$

we find the two terms

$$(A(2) \otimes_1 (C(1) \otimes_2 C()1)) \quad \text{and} \quad (B(2)) \otimes_1 ((D(1) \otimes_2 D(1)))$$

in two separate coproducts which are joined by $\otimes_3$.

The map ξ^{12} thus uses first η^{23}, then η^{13} and finally the hypothesis that $(\mathcal{V}, \coprod, \otimes_3, \dots, \otimes_n)$ is an $(n-1)$-fold monoidal category; specifically instances of the map $(X \otimes_3 Y) \coprod (Z \otimes_3 W)$ to $(X \coprod Z) \otimes_3 (Y \coprod W)$.

It is also not hard to check the unit conditions which are required for the fibrewise products to be the multiplication for a monoid in the category of monoidal categories. The extra requirement of the two sorts of unit is that $\mathcal{M}\hat{\otimes}_1\mathcal{M} = \mathcal{M}$ and that $\mathcal{I}\hat{\otimes}_k\mathcal{I} = \mathcal{I}$ for $k > 1$. These equations do indeed hold. Thus the first product together with any of the fibrewise products are those of a 2-fold monoidal category.

For the products $\hat{\otimes}_2$ and higher the associators and interchange transformations are fibrewise and the axioms hold since they hold for each fiber. Lastly we need to mention that the giant hexagon diagram for $i, j, k = 1, j, k$ commutes. This can be seen by splitting the hexagon into two commuting diagrams, one made up of the fibrewise applied interchangers of $\mathcal{V}$ and another that is a giant hexagon in the $(n-1)$-fold monoidal category $(\mathcal{V}, \coprod, \otimes_{m+1}, \dots, \otimes_n)$. □

3.5. *Remark.* In the context of [3] the lax functoriality of the tensor product with respect to the coproduct is due to the hypothesis that symmetric $\mathcal{V}$ is closed (from the right) with respect to the tensor product. This guarantees that that product preserves colimits on the first operand, since the functor $(\text{—} \otimes B)$ has as a right adjoint the internal hom, denoted by $[B, \text{—}]$. Applied to the coproduct this fact in turn implies that there is a canonical map in $\mathcal{V}$ from $(A \otimes B) \coprod (C \otimes D)$ to $(A \coprod C) \otimes (B \coprod D)$. From the universal properties of the coproduct it can be checked that this map satisfies the the middle interchange law that is required of a monoidal functor. Also in [3] Batanin points out that a fibrewise product is a monoidal functor with respect to the collection product. In that paper the existence of the transformation ξ depends on the symmetry (braiding) and the lax functoriality of the tensor product with respect to the coproduct. In this paper we chose to simply include the necessary iterated monoidal structure as a hypothesis, rather than the hypothesis of closedness, in the interest of generality.

Theorem 3.4 is quite useful for describing n-fold operads and their higher-categorical structure, especially when coupled with two other facts. The first is that monoids are equivalently defined as single object enriched categories, and the second is the following result from [11] and [12]. In those sources the quantifier lax is sometimes left off, but the proofs in question nowhere require the associator to be an isomorphism.

3.6. **Theorem.** *For $\mathcal{V}$ n-fold (lax) monoidal the category of enriched categories over $(\mathcal{V}, \otimes_1)$ is an $(n-1)$-fold monoidal 2-category.*

For our purposes we translate the theorem about enriched categories into its single object corollary about the category $\mathrm{Mon}(\mathcal{V})$ of monoids in $\mathcal{V}$.

3.7. **Corollary.** *For $\mathcal{V}$ n-fold (lax) monoidal, the category $\mathrm{Mon}(\mathcal{V})$ is an $(n-1)$-fold monoidal 2-category.*

Proof. The product of enriched categories always has as its object set the cartesian product of the object sets of its components. Thus one object enriched categories have products with one object as well. □

3.8. **Definition.** If an n-fold monoidal category $\mathcal{V}$ has coproducts and $(\mathcal{V}, \coprod, \otimes_3, \dots, \otimes_n)$ is an $(n-1)$-fold monoidal category in which each of the functors $(— \otimes_i A)$ and $(A \otimes_i —)$ preserves coproducts we define the category of m-fold operads $\mathrm{Oper}_m(\mathcal{V})$ to be the category of monoids in the category of collections $(\mathrm{Col}_m(\mathcal{V}), \hat{\otimes}_1)$ for $n \geq m \geq 2$.

3.9. **Corollary.** $\mathrm{Oper}_m(\mathcal{V})$ *is an $(n-m)$-fold monoidal 2-category.*

Proof. Rather than starting with monoids in n-fold monoidal $\mathcal{V}$ as in the previous corollary we are actually beginning with monoids in $(n-m+1)$-fold monoidal $\mathrm{Col}_m(\mathcal{V})$. Note that in [11] the products in $\mathcal{V}$ are assumed to have a common unit. To generalize to our situation here, where the unit for the first product in the category of collections is distinct, we need to add slightly to the definitions in [11]. When enriching (or more specifically taking monoids) we are doing so with respect to the first available product. Thus the unit morphism for enriched categories has its domain the unit for that first product, $\mathcal{I}$. However the unit enriched category $\boldsymbol{\mathcal{I}}$ has one object, denoted 0, and $\boldsymbol{\mathcal{I}}(0,0) = \mathcal{M}$. □

3.10. *Remark.* This theorem justifies our focus on the first m products of $\mathcal{V}$ as opposed to any subset of the n products. It is due to the way in which this focus allows us to describe the resulting structure on the category of m-fold operads. Of course, we can use the forgetful functors mentioned in Section 2 to pass from n-fold monoidal $\mathcal{V}$ to $\mathcal{V}$ with any of the subsets of products. The m-fold operads do behave as expected under this forgetting, retaining all but the structure that depends on the forgotten products. This will be seen more clearly upon inspection of the unpacked definition to follow. In short, we will see that an m-fold operad is also an $(m-1)$-fold operad.

3.11. *Remark.* We note that since a symmetric monoidal category is n-fold monoidal for all n, then operads in a symmetric monoidal category are n-fold monoidal for all n as well.

More generally, if $n \geq 3$ and the interchanges are isomorphisms, then by the Eckmann-Hilton argument the products collapse into one and the result is a a symmetric monoidal category, and so operads in it are again n-fold monoidal for all n. Here we are always discussing ordinary "non-symmetric," ("non-braided") operads. The possible faithful actions of symmetry or braid groups can be considered after the definition, which we leave for a later paper. We do point out that the proper direction in which to expand this work is seen in Weber's paper [30]. He generalizes by making a distinction between the binary and k-ary products in the domain of the composition map $\gamma\colon \mathcal{C}(k)\otimes(\mathcal{C}(j_1)\otimes\cdots\otimes\mathcal{C}(j_k))\to\mathcal{C}(j)$. The binary tensor product is seen formally as a pseudo-monoid structure and the k-ary product as a pseudo-algebra structure for a 2-monad which can contain the information needed to describe actions of braid or symmetry groups. The two structures are defined using strong monoidal morphisms, and so the products coincide and give rise to the braiding which is used to describe the associativity of composition. To encompass the definitions in this paper we would move to operads in lax-monoidal pseudo algebras, where instead of pseudo monoids and strong monoidal morphisms in a pseudo algebra we would consider the same picture but with lax monoidal morphisms.

The fact that monoids are single object enriched categories also leads to an efficient expanded definition of m-fold operads in an n-fold monoidal category. Let $\mathcal{V}$ be an n-fold monoidal category as defined in Section 2.

3.12**. Definition.** For $2\leq m\leq n$ an m-fold operad $\mathcal{C}$ in $\mathcal{V}$ consists of objects $\mathcal{C}(j)$, $j\geq 0$, a unit map $\mathcal{J}: I\to\mathcal{C}(1)$, and composition maps in $\mathcal{V}$

$$\gamma^{pq}:\mathcal{C}(k)\otimes_p(\mathcal{C}(j_1)\otimes_q\cdots\otimes_q\mathcal{C}(j_k))\to\mathcal{C}(j)$$

for $m\geq q>p\geq 1$, $k\geq 1$, $j_s\geq 0$ for $s=1\dots k$ and $\sum_{n=1}^{k} j_n = j$. The composition maps obey the following axioms:

(1) Associativity: The following diagram is required to commute for all $m\geq q>p\geq 1$, $k\geq 1$, $j_s\geq 0$ and $i_t\geq 0$, and where $\sum_{s=1}^{k} j_s=j$ and $\sum_{t=1}^{j} i_t = i$. Let $g_s=\sum_{u=1}^{s} j_u$ and let $h_s=\sum_{u=1+g_{s-1}}^{g_s} i_u$.

The η^{pq} labelling the leftmost arrow actually stands for a variety of equivalent maps which factor into instances of the pq interchange.

$$\begin{tikzcd}
\mathcal{C}(k)\otimes_p\left(\bigotimes_{s=1}^{k}{}_q\mathcal{C}(j_s)\right)\otimes_p\left(\bigotimes_{t=1}^{j}{}_q\mathcal{C}(i_t)\right) \arrow[r, "\gamma^{pq}\otimes_p\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes_p\eta^{pq}"'] & \mathcal{C}(j)\otimes_p\left(\bigotimes_{t=1}^{j}{}_q\mathcal{C}(i_t)\right) \arrow[d, "\gamma^{pq}"] \\
 & \mathcal{C}(i) \\
\mathcal{C}(k)\otimes_p\left(\bigotimes_{s=1}^{k}{}_q\mathcal{C}(j_s)\otimes_p\left(\bigotimes_{u=1}^{j_s}{}_q\mathcal{C}(i_{u+g_{s-1}})\right)\right) \arrow[r, "\mathrm{id}\otimes_p(\otimes_q^k\gamma^{pq})"'] & \mathcal{C}(k)\otimes_p\left(\bigotimes_{s=1}^{k}{}_q\mathcal{C}(h_s)\right) \arrow[u, "\gamma^{pq}"']
\end{tikzcd}$$

(2) Respect of units is required just as in the symmetric case. The following unit diagrams commute for all $m \geq q > p \geq 1$.

$$\begin{tikzcd}
\mathcal{C}(k) \otimes_p (\otimes_q^k I) \arrow[r, equal] \arrow[d, "1\otimes_p(\otimes_q^k \mathcal{J})"'] & \mathcal{C}(k) \\
\mathcal{C}(k) \otimes_p (\otimes_q^k \mathcal{C}(1)) \arrow[ur, "\gamma^{pq}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes_p \mathcal{C}(k) \arrow[r, equal] \arrow[d, "\mathcal{J}\otimes_p 1"'] & \mathcal{C}(k) \\
\mathcal{C}(1) \otimes_p \mathcal{C}(k) \arrow[ur, "\gamma^{pq}"] &
\end{tikzcd}$$

3.13. **Theorem.** *The description of m-fold operad in Definition 3.12 is equivalent to that given in Definition 3.8.*

Proof. If a collection has an operad composition $\gamma^{q,q+1}$ using $\otimes_q$ and $\otimes_{q+1}$ then it automatically has an operad composition for any pair of products $\otimes_p$ and $\otimes_s$ for $p < s \leq q+1$. This follows from the fact that for $p < q$ we have natural transformations $\eta^{pq}_{AIIB} : A \otimes_p B \to A \otimes_q B$, as described at the end of Definition 2.1. Thus if we have $\gamma^{q,q+1}$ then we can form $\gamma^{ps} = \gamma^{q,q+1} \circ (\eta^{pq} \circ (1 \otimes_q \eta^{s,q+1}))$. The new γ^{ps} is associative based on the old γ's associativity, the naturality of η, and the coherence of η. Thus follows our claim that generally operads are preserved as such by the forgetful functors mentioned in Section 2 and specifically that an m-fold operad is also an $(m-1)$-fold operad. The converse of this latter statement is not true, as we will see by counterexample in the last section. It will demonstrate the existence of m-fold operads which are not $(m+1)$-fold operads. □

It is also worth while to expand the definition of the tensor products of m-fold operads that is implicit in their depiction as monoids in the category of collections in an n-fold monoidal category. Here is the expanded version of the definition:

3.14. **Definition.** Let $\mathcal{C}$,$\mathcal{D}$ be m-fold operads. For $1 \leq i \leq (n-m)$ and using a $\otimes'_k$ to denote the product of two m-fold operads, we define that product to be given by:

$$(\mathcal{C} \otimes'_i \mathcal{D})(j) = \mathcal{C}(j) \otimes_{i+m} \mathcal{D}(j).$$

We note that the new γ is in terms of the two old ones, for $m \geq q > p \geq 1$:

$$\gamma^{pq}_{\mathcal{C}\otimes'_i\mathcal{D}} = (\gamma^{pq}_{\mathcal{C}} \otimes_{i+m} \gamma^{pq}_{\mathcal{D}}) \circ \eta^{p,i+m} \circ (1 \otimes_p \eta^{q,i+m})$$

where the subscripts denote the n-fold operad the γ belongs to and the η's actually stand for any of the equivalent maps which factor into them. Note that this expansion also helps make clear why it is that the monoidalness, or number of products, of m-fold operads must decrease by the same number m. From the condensed version this is expected due to the iterated enrichment. From the expanded view this allows us to define the new composition since in order for the products of operads to be closed, γ for the i^{th} product utilizes an interchange with superscript $i+m$. Defined this way i can only be allowed to be as large as $n-m$. We demonstrate in the last section in fact a counterexample which shows that the degree of monoidalness for the category of m-fold operads in an n-fold monoidal category is in general no greater than $n-m$.

4. Examples of iterated monoidal categories

4.1. **Lemma.** *Given a totally ordered set S with a least element $e \in S$, then the elements of S make up the objects of a strict monoidal category.*

The category will also be denoted S. Its morphisms are given by the ordering; there is only an arrow $a \to b$ if $a \leq b$. The product is max and the 2-sided unit is the least element e. We must check that the product is functorial since this defines monoidal structure on morphisms. Here it is so since if $a \leq b$ and $a' \leq b'$ then $\max(a, a') \leq \max(b, b')$. Also the identity is clearly preserved.

4.2. **Example.** The basic example is the nonnegative integers $\mathbb{N}$ with their ordering $\leq$.

4.3. **Lemma.** *Any ordered monoid with its identity element e also its least element forms the object set of a 2-fold monoidal category.*

Proof. Morphisms are again given by the ordering. The products are given by max and the monoid operation: $a \otimes_1 b = \max(a, b)$ and $a \otimes_2 b = ab$. The shared two-sided unit for these products is the identity element 0. The products are both strictly associative and functorial since if $a \leq b$ and $a' \leq b'$ then $aa' \leq bb'$ and $\max(a, a') \leq \max(b, b')$. The interchange natural transformations exist since $\max(ab, cd) \leq \max(a, c) \max(b, d)$. That is because

$$a \leq \max(a, c) \quad \text{and} \quad b \leq \max(b, d)$$

so

$$ab \leq \max(a, c) \max(b, d) \quad \text{and} \quad cd \leq \max(a, c) \max(b, d)$$

The internal and external unit and associativity conditions of Definition 2.1 are all satisfied due to the fact that there is only one morphism between two objects. More generally, given any ordered n-fold monoidal category with I the least object we can potentially form an $(n+1)$-fold monoidal category with morphisms ordering, and the new $\otimes_1 = \max$. □

4.4. **Example.** Again we have in mind $\mathbb{N}$ with its ordering and addition.

Other examples of such monoids as in Lemma 4.3 are the pure braids on n strands with only right-handed crossings [16]. Notice that braid composition is a non-symmetric example. Further examples are found in the papers on semirings and idempotent mathematics, such as [18] and its references as well as on the related concept of tropical geometry, such as [25] and its references. Semirings that arise in these two areas of study usually require some translation of the lemmas we have stated thus far, since the idempotent operation is usually min and its unit ∞. Also, since the operation given by addition has unit 0, we have to broaden our definition of 2-fold monoidal category. Working from the principle that the second operation is the multiplication of a categorical monoid with respect to the first, the additional requirement is that the two distinct units obey each other's operations: i.e $I_1 \otimes_2 I_1 = I_1$ and $I_2 \otimes_1 I_2 = I_2$. For example, $\min(0, 0) = 0$ and $\infty + \infty = \infty$.

4.5. **Example.** If S is an ordered set then by $\mathrm{Seq}(S)$ we denote the infinite sequences X_n of elements of S for which there exists a natural number $l(X)$ called the length such that $k > l(X)$ implies $X_k = e$ and $X_{l(X)} \neq e$. Under lexicographic ordering $\mathrm{Seq}(S)$ is in turn a totally ordered set with a least element. The latter is the sequence 0 where $0_n = e$ for all n. We let $l(0) = 0$. The lexicographic order means that $A \leq B$ if either $A_k = B_k$ for all k or there is a natural number $n = n_{AB}$ such that $A_k = B_k$ for all $k < n$, and such that $A_n < B_n$.

The ordering is easily shown to be reflexive, transitive, and antisymmetric. See for instance [23] where the case of lexicographic ordering of n-tuples of natural numbers is discussed. In our case we will need to modify the proof given in that source by always making comparisons of $\max(l(A), l(B))$-tuples.

As a category $\mathrm{Seq}(S)$ is 2-fold monoidal since we can demonstrate two interchanging products. They are max using the lexicographic order: $A \otimes_1 B = \max(A, B)$; and concatenation of sequences:

$$(A \otimes_2 B)_n = \begin{cases} A_n, & n \le l(A) \\ B_n, & n > l(A) \end{cases}$$

Concatenation clearly preserves the ordering.

4.6**. Example.** Letting S be the set with a single element recovers Example 4.4 as $\mathrm{Seq}(S)$.

4.7**. Lemma.** *If we have an ordered monoid* $(M, +)$ *as in Lemma 4.3 and reconsider* $\mathrm{Seq}(M)$ *as in Example 4.5 then we can describe a 3-fold monoidal category* $\mathrm{Seq}(M, +)$ *(with* $\mathrm{Seq}(M)$ *the image of forgetting the third product of pointwise addition) iff the monoid operation* $+$ *is such that* $0 < a < b$ *and* $c \le d$ *imply both* $a + c < b + d$ *and* $c + a < d + b$ *strictly.*

Proof. The first two products are again lexicographic max and concatenation of sequences. The third product $\otimes_3$ is pointwise application of $+$, $(A \otimes_3 B)_n = A_n + B_n$. The last condition that the monoid operation $+$ strictly respect strict ordering is necessary to guarantee that the third product both respect the lexicographic ordering; and interchange correctly with concatenation. To see the former let sequences $A \le B, C \le D$. Note that if $A = B, C = D$ then $A \otimes_3 C = B \otimes_3 D$ and if instead (without loss of generality) $A_j < B_j$ for j such that $A_i = B_i$ and $C_i = D_i$ for $i \le j$, then $A \otimes_3 C < B \otimes_3 D$, since $C_j \le D_j$. To see the converse, consider a case where $0 < a < b$ and $c \le d$ but $a + c = b + d$. Then the sequences $A = (a, a)$, $B = (b, 0)$, $C = (c, 0)$, $D = (d, 0)$ are such that lexicographically $A < B$ and $C \le D$ but $A \otimes_3 C = (a + c, a) > B \otimes_3 D = (b + d, 0)$. To see the interchange $(A \otimes_3 B) \otimes_2 (C \otimes_3 D) \le (A \otimes_2 C) \otimes_3 (B \otimes_2 D)$ notice that we can assume that $l(A) > l(B)$. Then

$$\mathrm{Concat}(A + B, C + D) \le \mathrm{Concat}(A, C) + \mathrm{Concat}(B, D)$$

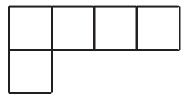
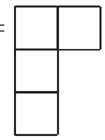
due to the fact that if D has a first non-zero term, it will be added to an earlier term of the concatenation of A and C in the second four-fold product. □

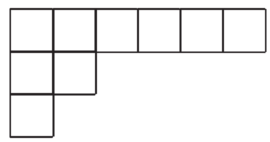
4.8*. Remark.* A non-example is seen if we begin with the monoid of Lemma 4.1, an ordered set with a least element where the product is max. Here max does not strictly preserve strict ordering, and so pointwise max does not respect lexicographic ordering. Neither do concatenation and pointwise max interchange.

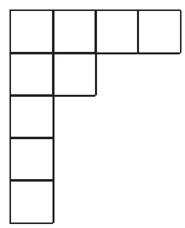
4.9**. Corollary.** *Given any ordered* n*-fold monoidal category* C *with* I *the least object and* $\otimes_1$ *the max, and whose higher products strictly respect strict ordering, we can form an* $(n+1)$*-fold monoidal category* $\mathrm{Seq}(C)$*.*

Proof. The new products of $\mathrm{Seq}(C)$ are the lexicographic max, the concatenation, and the pointwise application of $\otimes_i$ for $i = 2 \dots n$. The pointwise application of the original products to the sequences directly inherits the interchange properties. For instance, if $A, B, C, D \in \mathrm{Seq}(C)$ then $(A_n \otimes_2 B_n) \otimes_1 (C_n \otimes_2 D_n) \le (A_n \otimes_1 C_n) \otimes_2 (B_n \otimes_1 D_n)$ for all n, which certainly implies that the pointwise 4-fold products are ordered lexicographically. □

4.10**. Example.** Even more symmetrical structure is found in examples with a natural geometric representation which allows the use of addition in each product. One such category is that whose objects are Young diagrams, by which we mean the underlying shapes or diagrams of Young tableaux. These can be presented by a decreasing sequence of nonnegative integers in two ways: the sequence that gives the heights of the columns or the sequence that gives the lengths of the rows. We let $\otimes_2$ be the product which adds the heights of columns of two diagrams, $\otimes_1$ adds the length of rows. We often refer to these as vertical and horizontal stacking respectively. If

$A =$  and $B =$ 

then $A \otimes_1 B =$ 

and $A \otimes_2 B =$ 

There are several possibilities for morphisms. We can take as morphisms the totally ordered structure of the Young diagrams given by lexicographic ordering. In interest of focusing on the stacking products though we may choose to restrict these morphisms further, and say an arrow given by ordering can only exist between similar mass objects, i.e. the two objects in question have equal sums of their respective sequences or, in reference to the pictures, an equal total number of blocks. This restriction eliminates the product described by lexicographic max. By the category of restricted Young diagrams, we will refer to morphisms as restricted lexicographic ordering, and the two stacking products demonstrated above. We will often find occasion to relax the morphisms to include all ordering and reintroduce the lexicographic max as $\otimes_1$, and will refer to that category simply as the category of Young diagrams.

By previous discussion of sequences the Young diagrams with $\otimes_1$ the lexicographic max and $\otimes_3$ the piecewise addition (thought of here as vertical stacking) form a subcategory of the 3-fold monoidal category called $\mathrm{Seq}(\mathbb{N}, +)$. To see that with the additional $\otimes_2$ of horizontal stacking that this becomes a valid 3-fold monoidal category we look at that operation from another point of view. Note that the horizontal product of Young diagrams A and C can be described as a reorganization of all the columns of both A and C into a new Young diagram made up of those columns in descending order of height. Rather than (but equivalent to) the addition of rows, we see horizontal stacking as the concatenation of monotone decreasing sequences (of columns) followed by sorting greatest to least. We call this operation merging.

4.11**. Lemma.** *Let* $(S, \leq, +)$ *be an ordered monoid and consider the sequences* $\mathrm{Seq}(S, +)$ *with lexicographic ordering, piecewise addition* $+$ *and the function of sorting denoted by*

$$s\colon \mathrm{Seq}(S, +) \to \mathrm{Seq}(S, +)$$

Then the triangle inequality holds for two sequences: $s(A+B) \leq s(A)+s(B)$.

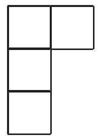
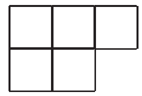
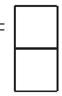
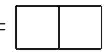
Proof. Consider $s(A+B)$, where we start with the two sequences and add them piecewise before sorting. We can metamorphose this into $s(A)+s(B)$ in stages by using an algorithm to sort A and B. Note that if A and B are already sorted, the inequality becomes an equality. For our algorithm we choose parallel bubble sorting. This consists of a series of passes through the sequences comparing A_n and A_{n+1} and comparing B_n and B_{n+1} simultaneously. If the two elements of a given sequence are not already in strictly decreasing order we switch their places. We claim that switching consecutive sequence elements into order always results in a lexicographically larger sequence after adding piecewise and sorting. If both the elements of A and of B are switched, or if neither, then the result is unaltered. Therefore without loss of generality we assume that $A_n < A_{n+1}$ and that $B_{n+1} < B_n$. Then we compare the original result of sorting after adding and the same but after the switching of A_n and A_{n+1}. It is simplest to note that the new result includes $A_{n+1}+B_n$, which is larger than both A_n+B_n and $A_{n+1}+B_{n+1}$. So after adding and sorting the new result is indeed larger lexicographically. Thus since each move of the parallel bubble sort results in a larger expression after first adding and then sorting, and after all the moves the result of adding and then sorting the now presorted sequences is the same as first sorting then adding, the triangle inequality follows. ☐

4.12**. Theorem.** *The category of restricted Young diagrams forms a 2-fold monoidal category, and the category of Young diagrams forms a 3-fold monoidal category.*

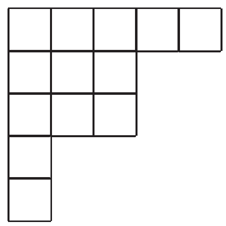
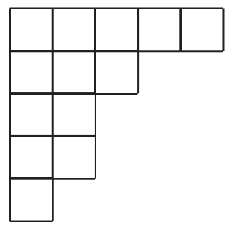
Proof. We show the latter statement is true, and then note that the the former statement follows since the restricted Young diagrams are just the image of forgetting the first product on Young diagrams and then passing to a subcategory by restricting morphisms. The products on Young diagrams are $\otimes_1$ = lexicographic max, $\otimes_2$ = horizontal stacking and $\otimes_3$ = vertical stacking. We need to check first that horizontal stacking, or merging, is functorial with respect to morphisms (defined as the $\leq$ relations of the lexicographic ordering.) The cases where $A = B$ or $C = D$ are easy. For example let $A_k = B_k$ for all k and $C_k = D_k$ for all $k < n_{CD}$, where n_{CD} is as defined in Example 4.5. Thus the columns from the copies of, for instance A in $A \otimes_1 C$ and $A \otimes_1 D$ fall into the same final spot under the sortings right up to the critical location, so if $C \leq D$, then $A \otimes_1 C \leq A \otimes_1 D$. Similarly, it is clear that $A \leq B$ implies $(A \otimes_1 D) \leq (B \otimes_1 D)$. Hence if $A \leq B$ and $C \leq D$, then $A \otimes_1 C \leq A \otimes_1 D \leq B \otimes_1 D$ which by transitivity gives us our desired property.

Next we check that our interchange transformations will always exist. η^{1j} exists by the proof of Lemma 4.3 for $j = 2, 3$ since the higher products both respect morphisms(ordering) and are thus ordered monoid operations. We need to check for existence of η^{23}, i.e. we need to show that $(A \otimes_3 B) \otimes_2 (C \otimes_3 D) \leq (A \otimes_2 C) \otimes_3 (B \otimes_2 D)$. This inequality follows from Lemma 4.11 on the triangle inequality for sorting. To prove the new inequality we consider the special case of two sequences formed by letting A' be A followed by C and letting B' be B followed by D. By "followed by" we mean padded by zeroes so that $l(A') = \max(l(A), l(B)) + l(C)$ and $l(B') = \max(l(A), l(B)) + l(D)$. Thus piecewise addition of A' and B' results in piecewise addition of A and B, and respectively C and D. Then to our new sequences A' and B' we apply the result of Lemma 4.11 and have our desired result. ☐

Here is an example of the inequality we have just shown to always hold. Let four Young diagrams be as follow:

$A =$  $B =$  $C =$  $D =$ 

Then the fact that $(A \otimes_3 B) \otimes_2 (C \otimes_3 D) \leq (A \otimes_2 C) \otimes_3 (B \otimes_2 D)$ appears as follows:

 $\leq$ 

4.13. *Remark.* Alternatively we can create a category equivalent to the non-negative integers in Example 4.2 by pre-ordering the Young diagrams by height. Here the height $h(A)$ of the Young diagram is the number of boxes in its leftmost column, and we say $A \leq B$ if $h(A) \leq h(B)$. Two Young diagrams with the same height are isomorphic objects, and the one-column stacks form both a full subcategory and a skeleton of the height preordered category. Everything works as for the previous example of natural numbers since $h(A \otimes_2 B) = h(A) + h(B)$ and $h(A \otimes_1 B) = \max(h(A), h(B))$. There is also a max product; the new max with respect to the height preordering is defined as

$$\max(A, B) = \begin{cases} A, & \text{if } B \leq A \\ B, & \text{otherwise.} \end{cases}$$

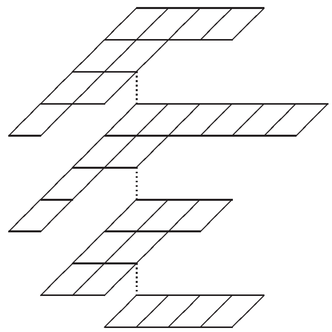
In the height preordered category this latter product is equivalent to horizontal stacking, $\otimes_1$.

4.14. *Remark.* Notice that we can start with any totally ordered monoids $\{M, \leq, +\}$ such that the identity 0 is less than any other element and such that $0 < a < b$ and $c \leq d$ implies both $a + c < b + d$ and $c + a < d + b$ for all $a, b, c \in G$. We create a 3-fold monoidal category $\mathrm{ModSeq}(M, +)$ with objects monotone decreasing finitely non-zero sequences of elements of M and morphisms given by the lexicographic ordering. The products are as described for the category of Young diagrams $\mathrm{ModSeq}(\mathbb{N}, +)$ in the previous example. The common unit is the zero sequence. The proofs we have given in the previous example for $M = \mathbb{N}$ are all still valid.

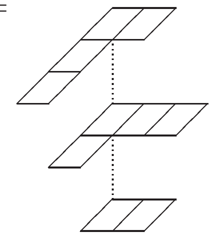
By Corollary 4.9 we can also consider 4-fold monoidal categories such as $\mathrm{Seq}(\mathrm{ModSeq}(M))$ and other combinations of Seq and ModSeq. For instance if $\mathrm{ModSeq}(\mathbb{N}, +)$ is our category of Young diagrams then $\mathrm{ModSeq}(\mathrm{ModSeq}(\mathbb{N}, +))$ has objects monotone decreasing sequences of Young diagrams, which we can visualize along the z-axis. Here the lexicographic-lexicographic max is $\otimes_1$, lexicographic merging is $\otimes_2$, pointwise merging (pointwise horizontal or y-axis stacking) is $\otimes_3$ and pointwise-pointwise addition (pointwise x-axis stacking) is $\otimes_4$.

For example, if:

$A =$

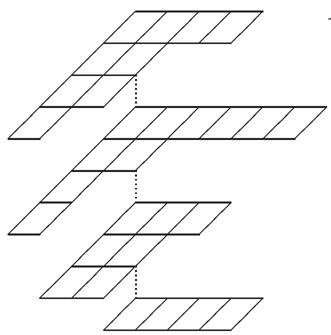


and $B =$

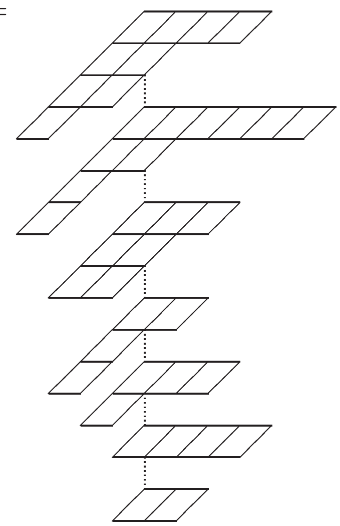


then

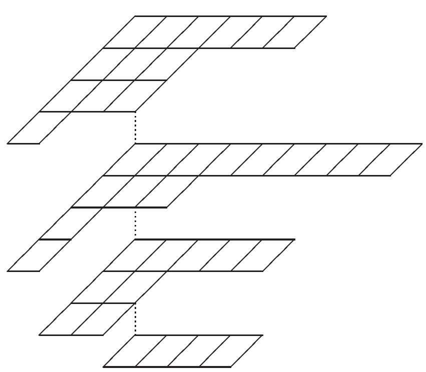
$A \otimes_1 B =$



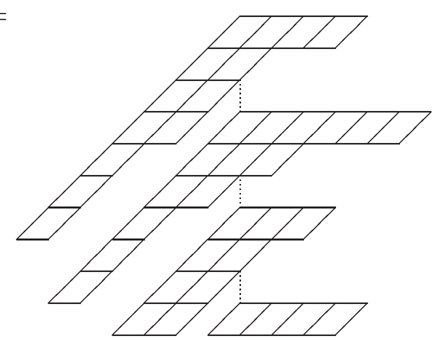
$A \otimes_2 B =$



$A \otimes_3 B =$



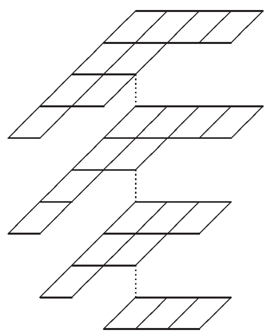
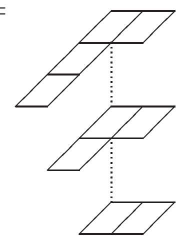
and $A \otimes_4 B =$



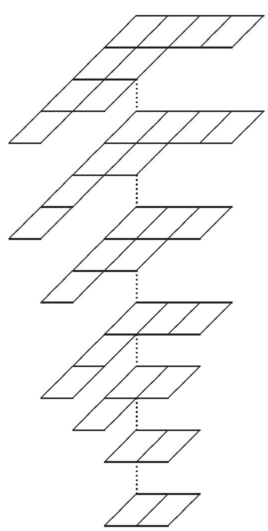
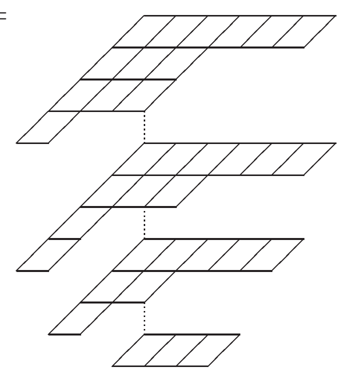
4.15**. Example.** It might be nice to retain the geometric picture of the products of Young diagrams in terms of vertical and horizontal stacking, and stacking in other directions as dimension increases. This is not found in the just illustrated category, which relies on the merging viewpoint. The "diagram stacking" point of view is restored if we restrict to 3-d Young diagrams. We can represent these objects as infinite matrices with finitely nonzero natural number entries, and with monotone decreasing columns and rows. We require that

A_{nk} be decreasing in n for constant k, and decreasing in k for constant n. We choose the sequence of rows to represent the sequence of sequences, i.e. each row represents a Young diagram which we draw as being parallel to the xy plane. This choice is important because it determines the total ordering of matrices and thus the morphisms of the category. Thus y-axis stacking is horizontal concatenation (disregarding trailing zeroes) of matrices followed by sorting the new longer rows (row merging). x-axis stacking is addition of matrices. Now we define z-axis stacking as vertical concatenation of matrices followed by sorting the new long columns (column merging).

Here is a visual example of the three new products, beginning with z-axis stacking, labeled $\otimes_1$: if

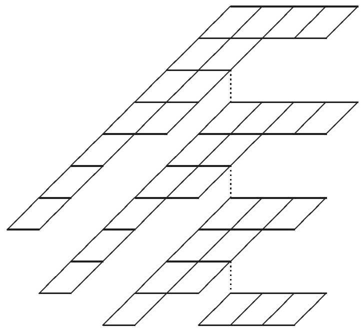
$A =$  and $B =$ 

then we let

$A \otimes_1 B =$  , $A \otimes_2 B =$ 

and

$A \otimes_3 B =$



Note that in this restricted setting of decreasing matrices the lexicographic merging of sequences (rows) of two matrices does not preserve the total decreasing property (decreasing in rows and columns).

These three products just shown preserve the total sum of the entries in both matrices, and do interact via interchanges to form the structure of a 3-fold category. Renumbered, they are: $\otimes_1$ (z-axis stacking) is the vertical concatenation of matrices followed by sorting the new longer columns, $\otimes_2$ (y-axis stacking) is horizontal concatenation of matrices followed by sorting the new longer rows and $\otimes_3$ (x-axis stacking) is the addition of matrices. For comparison, here is the same example of the products as just given above shown by matrices. Only the non-zero entries of the matrices are shown.

$$A = \begin{bmatrix} 4 & 3 & 1 & 1 \\ 4 & 2 & 1 & 1 \\ 3 & 2 & 1 & \\ 1 & 1 & 1 & \end{bmatrix} B = \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A \otimes_1 B = \begin{bmatrix} 4 & 3 & 1 & 1 \\ 4 & 2 & 1 & 1 \\ 3 & 2 & 1 & \\ 3 & 1 & 1 & \\ 2 & 1 & & \\ 1 & 1 & & \\ 1 & 1 & & \end{bmatrix} A \otimes_2 B = \begin{bmatrix} 4 & 3 & 3 & 1 & 1 & 1 \\ 4 & 2 & 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 & 1 & \\ 1 & 1 & 1 & & & \end{bmatrix} \text{ and } A \otimes_3 B = \begin{bmatrix} 7 & 4 & 1 & 1 \\ 6 & 3 & 1 & 1 \\ 4 & 3 & 1 & \\ 1 & 1 & 1 & \end{bmatrix}$$

4.16**. Theorem.** *The category of 3-d Young diagrams with lexicographic ordering and the products just described possesses the structure of a 3-fold monoidal category.*

Proof. We already have existence of η^{23} by the previous argument about pointwise application of two interchanging products. To show existence of $\eta^{13} : (A \otimes_3 B) \otimes_1 (C \otimes_3 D) \to (A \otimes_1 C) \otimes_3 (B \otimes_1 D)$ we need to check that sorting each of the columns of two pairs of vertically concatenated matrices before pointwise adding gives a larger lexicographic result with respect to rows than adding first and then sorting columns. This follows from Lemma 4.11, applied to each pair of sequences which are the n^{th} columns in the two new matrices formed by vertically concatenating A and C and respectively B and D, padded with zeroes so that adding the new matrices results in adding A and B and respectively C and D. From the lemma then we have that $(A \otimes_1 C) \otimes_3 (B \otimes_1 D)$ gives a result whose n^{th} column is lexicographically greater than or equal to the n^{th} column of $(A \otimes_3 B) \otimes_1 (C \otimes_3 D)$. This implies that either the pairs of respective columns are each equal sequences or that there is some least row i and column j such that all the pairs of columns are identical in rows less than i and that the two rows i are identical in columns less than j, but that the i,j position in $(A \otimes_3 B) \otimes_1 (C \otimes_3 D)$ is less than the corresponding position in $(A \otimes_1 C) \otimes_3 (B \otimes_1 D)$. Thus the existence of the required inequality is shown.

The existence of η^{12} is due to the fact that we are ordering the matrices by giving precedence to the rows. The two four-fold products can be seen as two alternate operations on a single large matrix M. This matrix is constructed by arranging A, B, C, D with added zeroes so that $(A \otimes_1 C) \otimes_2 (B \otimes_1 D)$ is the result of first sorting each column vertically, greater values at the top, and then each row horizontally, greater values to the left, while $(A \otimes_2 B) \otimes_1 (C \otimes_2 D)$ is achieved by sorting horizontally first and then vertically. Recall that in the ordering rows are given precedence over columns. Here is an illustration of the inequality, showing the process of constructing the large matrix.

$$A = \begin{bmatrix} 3 & 3 & 2 \\ 1 & 1 & \end{bmatrix} B = \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix} C = \begin{bmatrix} 2 \\ 1 \end{bmatrix} D = \begin{bmatrix} 5 \end{bmatrix}$$

$$M = \begin{bmatrix} 3 & 3 & 2 & 9 \\ 1 & 1 & 0 & 9 \\ 0 & 0 & 0 & 9 \\ 2 & 0 & 0 & 5 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$(A \otimes_2 B) \otimes_1 (C \otimes_2 D) = \begin{bmatrix} 9 & 3 & 3 & 2 \\ 9 & 2 & 1 & \\ 9 & 1 & & \\ 5 & & & \\ 1 & & & \end{bmatrix} < \begin{bmatrix} 9 & 3 & 3 & 2 \\ 9 & 2 & 1 & \\ 9 & 1 & & \\ 5 & 1 & & \end{bmatrix} = (A \otimes_1 C) \otimes_2 (B \otimes_1 D)$$

The proof that this inequality always holds requires the following two lemmas.

4.17. **Lemma.** *For two sequences of n elements each, the first given by $a_1 \dots a_n$ and the second by $b_1 \dots b_n$, then considering pairs of elements $a_{\sigma(i)}$ and $b_{\tau(i)}$ for permutations $\sigma, \tau \in S_n$, we have the following inequality:*

$$\max(\min(a_{\sigma(1)}, b_{\tau(1)}), \dots, \min(a_{\sigma(n)}, b_{\tau(n)})) \leq \min(\max(a_1, \dots, a_n), \max(b_1, \dots, b_n)).$$

This is true since for $i = 1 \dots n$ we have $a_i \leq \max(a_1, \dots, a_n)$ and $b_i \leq \max(b_1, \dots, b_n)$. Therefore $\min(a_{\sigma(i)}, b_{\tau(i)}) \leq \min(\max(a_1, \dots, a_n), \max(b_1, \dots, b_n))$ and the inequality follows.

4.18. **Lemma.** *For a given finite matrix M with n rows, we claim that first ordering each row (greater to lesser) and then sorting each resulting column gives a final result that is lexicographically less than or equal to the final result of sorting each column of M and then each row.*

This is seen by a chain of inequalities that each correspond to a single step in a parallel bubble sorting of the rows of M. Consider the final result of sorting each column vertically and then each row. We gradually evolve this into the reverse procedure by performing a series of steps, each of which begins by comparing two adjacent columns in the current stage of the evolution. The step consists of switches that insure each horizontal pair in the columns is in order, i.e. switching the positions of the two elements in each row only if the one in the left column is smaller than the one in the right. We call this a parallel switch, or just a switch. The result of taking the switched matrix and vertically sorting its columns and then horizontally sorting its rows will be shown to be lexicographically less than or equal to the result of vertically sorting columns and then horizontally sorting rows before the parallel switch. The entire series of steps together constitute sorting each row of M. Since after vertically sorting a matrix which began with sorted rows the new rows still remain sorted, then at the end of the evolution we are indeed doing the reverse procedure; that is sorting horizontally first and then vertically.

For a single step in the parallel bubble sort, we claim that after the parallel switch and then vertical sorting of the two adjacent columns the pairs in each resulting row will be either all identical to those in the result of vertically sorting the unswitched columns, or there will be a first row k in which the pair in the switched version of the columns consists of one element equal to one element of the corresponding pair in the unswitched version and one element less than the other element in the unswitched version.

Since no other columns are changed at this step, then this will imply that after vertically sorting the other columns and then all the rows in both matrices, the two resulting matrices will be identical or just identical up to the k^{th} row, where the switched matrix will be lexicographically less than the unswitched.

The claim for two columns follows from repeated application of Lemma 4.17. Let the two columns be $a_1 \dots a_n$ and $b_1 \dots b_n$ After the parallel switching, the left column holds the max of each pair and the right the min. Vertical sorting moves the max of each column to the top row, and leaves all the new rows (of two elements each) still sorted left to right. Located in the left position of the new top row is

$$\max(\max(a_1, b_1), \dots, \max(a_n, b_n)) = \max(\max(a_1, \dots, a_n), \max(b_1, \dots, b_n))$$

the latter of which is the in the top row of the vertically sorted unswitched columns. The right position in the top row of the switched columns is

$$\max(\min(a_1, b_1), \dots, \min(a_n, b_n)),$$

which is less than or equal to

$$\min(\max(a_1, \dots, a_n), \max(b_1, \dots, b_n)),$$

the other element in the top row of the vertically sorted unswitched columns, by the preceeding lemma (with trivial permutations). If less than, then we are done. If equal then we note that the remaining rows $2 \dots n$ contain the same collection of elements a_i and b_i in both the switched and unswitched columns, i.e. we may assume that in vertically sorting either version we moved a_j and b_l to the top row. Note that since the rows in the switched version are sorted, $\max(a_l, b_l) \geq \min(a_j, b_j)$ and $\max(a_j, b_j) \geq \min(a_l, b_l)$. Thus the $\max(a_l, b_j)$ will always be in the left column and $\min(a_l, b_j)$ in the right.

Then the second row of the vertically sorted switched pair of columns is

$$\max(\max(a_1, b_1), \dots, \widehat{\max(a_j, b_j)}, \dots, \widehat{\max(a_l, b_l)}, \dots, \max(a_n, b_n), \max(a_l, b_j))$$

in the first position and

$$\max(\min(a_1, b_1), \dots, \widehat{\min(a_j, b_j)}, \dots, \widehat{\min(a_l, b_l)}, \dots, \min(a_n, b_n), \min(a_l, b_j))$$

in the second position, where the hats indicate missing elements. Whereas the second row of the vertically sorted unswitched columns is made up of

$$\max(\max(a_1, \ldots, \hat{a_j}, \ldots, a_n)) \text{ and } \max(\max(b_1, \ldots, \hat{b_l}, \ldots, b_n)).$$

Thus the left position in the second row of the switched version is the same value as one of the elements in the second row of the unswitched vertically sorted columns. By Lemma 4.17 with the evident permutations, the right position in the second row is less than or equal to the other element in the second row of the unswitched vertically sorted columns. If less than, then we are done, if equal then the process continues. If the 1^{st} through $(n-1)^{st}$ rows of the switched and unswitched columns contain the same values after vertical sorting, then so do the n^{th} rows.

This completes the proof of the lemma. By applying it to the large matrix M constructed of the four matrices A, B, C, D as described above, we have the proof of the theorem. □

Now we define the general n-fold monoidal category of n-dimensional Young diagrams. The proof of the theorem for three dimensions plays an important role in the general theorem, since each interchanger involves two products. Once we have decided to represent Young diagrams of higher dimension by arrays of natural numbers which decrease in each index, it is clear that each interchanger will either involve directly two of the indices of the array or one index as well as pointwise addition.

4.19**. Definition.** The category of n-dimensional Young diagrams consists of

(1) Objects $A_{i_1 i_2 \ldots i_{n-1}}$, finitely nonzero n-dimensional arrays of nonnegative integers which are monotone decreasing in each index, and
(2) Morphisms the order relations in the lexicographic ordering with precedence given to lesser indices.

There are n ways to take a product of two n-dimensional Young diagrams, which we visualize as arrays of natural numbers in $n-1$ dimensions. The products correspond to merging, i.e. concatenating and then sorting, in each of the $n-1$ possible directions, as well as pointwise addition as $\otimes_n$. The order of products is the reverse of the order of the indices. That is, for $k = 1 \ldots n-1$, $\otimes_k$ is merging in the direction of the index i_{n-k}.

4.20**. Theorem.** *The category of n-dimensional Young diagrams with the above products constitutes an n-fold monoidal category.*

Proof. We must show the existence of the interchangers η^{jk} as inequalities for $1 \le j < k \le n$. First we demonstrate the existence of the required inequality when $k < n$. For A, B, C, D n-dimensional Young diagrams seen as $(n-1)$-dimensional arrays, we let $M_{i_1 i_2 \ldots i_{n-1}}$ be a large array made by concatenating A and B in the direction of the index i_k, concatenating C and D in the direction of the index i_k, and then concatenating those two results in the direction of the index i_j. Zeros are added (see above for the two dimensional array example) so that the products $(A \otimes_k B) \otimes_j (C \otimes_k D)$ and $(A \otimes_j C) \otimes_k (B \otimes_j D)$ can then both be described as sorting $M_{i_1 i_2 \ldots i_{n-1}}$ in two directions; first i_k then i_j or vice versa respectively. That the inequality holds is seen as we compare the results position by position in the lexicographic order, i.e. reading lower indices first. The first differing value we come upon, say in location $(i_1 i_2 \ldots i_{n-1})$, then will necessarily be the first difference in the sub-array of two dimensions in the directions i_j and i_k determined by the location $(i_1 i_2 \ldots i_{n-1})$. Thus by the proof of

Lemma 4.18, the value in $(A \otimes_k B) \otimes_j (C \otimes_k D)$ is less than the corresponding value in $(A \otimes_j C) \otimes_k (B \otimes_j D)$.

Secondly we check the cases that have $k = n$. We can see the four-fold products as operations on two arrays, one made by concatenating A and C in the i_j direction, and another made by concatenating B and D in the i_j direction, padded with zeroes so that adding the two pointwise results in pointwise addition of A with B, and of C with D. Then $(A \otimes_k B) \otimes_j (C \otimes_k D)$ is adding first and then sorting in the i_j direction, while $(A \otimes_j C) \otimes_k (B \otimes_j D)$ is the reverse process. To see that the correct inequality holds we again compare the results position by position in lexicographic order. The first differeing value is also the first difference between the two coresponding 2 dimensional sub-arrays which are in the directions i_j and i_{n-1}. These sub-arrays are the results of sorting and then pointwise addition and vice versa respectively, and so by the proof for existence of η^{13} in Theorem 4.16 the desired result is shown. $\square$

5. Examples of n-fold operads

The categories from Section 4 give us a domain in which we can exhibit some concrete examples of operads. To have an operad with an element $\mathcal{C}(0)$ we will need to "compactify" by adjoining an object that is both initial and terminal to the example categories based on ordered monoids and sequences. This object we will denote by $\emptyset$ and the unique maps to $\emptyset$ will be called zero maps. Composition and tensor product with a zero map both yield a zero map. We define all products involving the object $\emptyset$ as an operand to be equal to $\emptyset$. $\emptyset$ will be designated the least object in the total order, except for the purpose of defining the product using max, as just stated. Thus the unique maps from $\emptyset$ will be the $\leq$ relations, and therefore the only way that two legs of a diagram will not commute is if one of them is a zero map and the other is not. In all the examples the composition is associative since it is based upon ordering, so all we need check for is the existence of that composition. Note that each of the following examples satisfy the hypothesis of Theorem 3.4 since taking the max is a coproduct. Thus all coproducts are certainly included among the objects, and the max is the first of the tensor products in the iterated monoidal structure. Also, max distributes over each $\otimes_i$ since each product preserves the ordering.

5.1. **Example.** Of course $\mathcal{C}(j) = \emptyset$ and $\mathcal{C}(j) = 0$ for all j are trivially operads, where 0 is the monoidal unit. First we look at the simplest interesting examples: 2-fold operads in an ordered monoid such as $\mathbb{N}$, where $\otimes_1$ is max and $\otimes_2$ is $+$. We always set $\mathcal{C}(0) = \emptyset$ but often only list the later terms. Note that the zero map cannot play the role of operad composition, since it will fail associativity. Therefore a 2-fold operad in $\mathbb{N}$ is a sequence $\mathcal{C}(j)$ of natural numbers which has the property that for any $j_1 \dots j_k$, $\max(\mathcal{C}(k), \sum \mathcal{C}(j_i)) \leq \mathcal{C}(\sum j_i)$ and for which $\mathcal{C}(1) = 0$. This translates into saying that for any two whole numbers x, y we have that $\mathcal{C}(x + y) \geq \mathcal{C}(x) + \mathcal{C}(y)$ and that $\mathcal{C}(1) = 0$. The latter condition both satisfies the unit axioms and makes it redundant to also insist that the sequence be monotone increasing. Perhaps the first example that comes to mind is the Fibonacci numbers. Minimal examples are formed by choosing a starting term or terms and then determining each later n^{th} term. For a starting finite sequence $0, a_2, \dots, a_l$ which obeys the the axioms of a 2-fold operad so far, the operad $\mathcal{C}_{0,a_2,\dots,a_l}$ is found by defining terms $\mathcal{C}_{a_1,\dots,a_l}(n)$ for $n > l$ to be the maximum of all the values of $\max(\mathcal{C}(k), \sum_{i=1}^{k} \mathcal{C}(j_i))$ where the sum of the j_i is n. Some basic examples are the following sequences.

$$\mathcal{C}_{0,1} = (0, 1, 1, 2, 2, 3, 3, \dots), \qquad \mathcal{C}_{0,0,1} = (0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3, \dots)$$
$$\mathcal{C}_{0,2} = (0, 2, 2, 4, 4, 6, 6, \dots), \qquad \mathcal{C}_{0,0,2} = (0, 0, 2, 2, 2, 4, 4, 4, 6, 6, 6, \dots)$$

and

$$\mathcal{C}_{0,1,2,4,8} = (0, 1, 2, 4, 8, 8, 9, 10, 12, 16, 16, 17, 18, 20, 24, \dots).$$

It is clear that the growth of these sequences oscillates around linear growth in a predictable way.

5.2**. Theorem.** *If "arbitrary" starting terms $0, a_2, \dots, a_k \in \mathbb{N}$ are given (themselves of course obeying the axioms of a 2-fold operad), then the n^{th} term of the 2-fold operad $\mathcal{C}_{0,a_2,\dots,a_k}$ in $\mathbb{N}$ is given by*

$$a_n = a_q + pa_k \text{ where } n = pk + q, \text{ for } p \in \mathbb{N}, 0 \le q < k.$$

Proof. We need to show that $\max(a_l, \sum_{i=1}^{l} a_{j_i})$ is always less than or equal to $a_q + pa_k$, where $n = pk + q$, for $p \in \mathbb{N}$, $0 \le q < k$. We need only consider the cases in which $l < n$. Since a_l is always included as one of the a_{j_i}, we need to show only that $\sum_{i=1}^{l} a_{j_i}$ is always less than or equal to $a_q + pa_k$ where the sum of the j_i is n. This follows by strong induction. The base case holds by definition. Let $j_i = p_i k + q_i$ for $p_i \in \mathbb{N}$, and $0 \le q_i < k$. Then $\sum q_i = n - k \sum p_i = pk + q - k \sum p_i < n$. Thus we have, by the growth property of operads and by induction:

$$\begin{aligned}
\sum_{i=1}^{l} a_{j_i} &= \sum a_{q_i} + a_k \sum p_i \\
&\le a_{(k(p - \sum p_i) + q)} + a_k \sum p_i \\
&= a_q + (p - \sum p_i) a_k + a_k \sum p_i \\
&= a_q + pa_k.
\end{aligned}$$

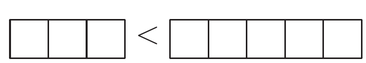
$\square$

5.3**. Example.** Consider the 3-fold monoidal category $\mathrm{Seq}(\mathbb{N}, +)$ of lexicographically ordered finitely nonzero sequences of the natural numbers (here we use $\mathbb{N}$ considered as an example of an ordered monoid), with products $\otimes_1$ the lexicographic max , $\otimes_2$ the concatenation and $\otimes_3$ the pointwise addition. An example of a 2-fold operad in $\mathrm{Seq}(\mathbb{N}, +)$ that is not a 3-fold operad is the following:

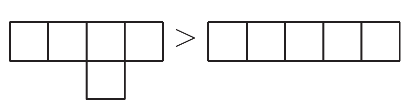
Let $\mathcal{B}(0) = \emptyset$ and let $\mathcal{B}(j)_i = 1$ for $i < j$, 0 otherwise. We can picture these as follows:

$$\mathcal{B}(1) = \text{———}\ , \ \mathcal{B}(2) = \square\ , \ \mathcal{B}(3) = \square\square\ , \ \mathcal{B}(4) = \square\square\square\ , \ \mathcal{B}(5) = \square\square\square\square\ , \ \dots$$

This is a 2-fold operad, with respect to the lexicographic max and concatenation. For instance the instance of composition $\gamma^{12} : \mathcal{B}(3) \otimes_1 (\mathcal{B}(2) \otimes_2 \mathcal{B}(1) \otimes_2 \mathcal{B}(3)) \to \mathcal{B}(6)$ appears as the relation:

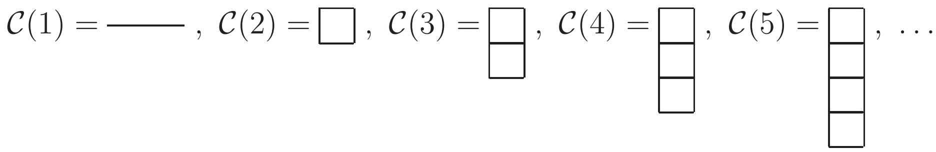


However, the relation

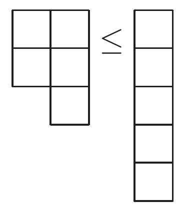


shows that $\gamma^{23} : \mathcal{B}(3) \otimes_2 (\mathcal{B}(1) \otimes_3 \mathcal{B}(3) \otimes_3 \mathcal{B}(2)) \to \mathcal{B}(6)$ does not exist, so that $\mathcal{B}$ is not a 3-fold operad.

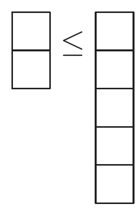
5.4**. Example.** Next we give an example of a 3-fold operad in $\mathrm{Seq}(\mathbb{N}, +)$. Let $\mathcal{C}(0) = \emptyset$ and let $\mathcal{C}(j) = (j-1, 0 \dots)$. We can picture these as follows:



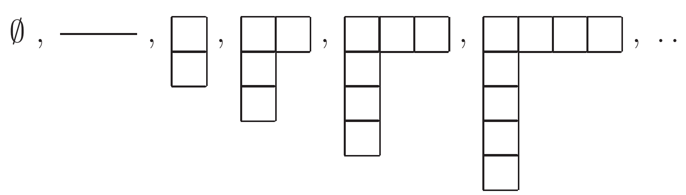
First we note that the operad $\mathcal{C}$ just given is a 3-fold operad since we have that the $\gamma^{23} : \mathcal{C}(k) \otimes_2 (\mathcal{C}(j_i) \otimes_3 \cdots \otimes_3 \mathcal{C}(j_k)) \to \mathcal{C}(j)$ exists. For instance $\gamma^{23} : \mathcal{C}(3) \otimes_2 (\mathcal{C}(1) \otimes_3 \mathcal{C}(3) \otimes_3 \mathcal{C}(2)) \to \mathcal{C}(6)$ appears as the relation



Then we remark that as expected the composition $\gamma^{12} : \mathcal{C}(k) \otimes_1 (\mathcal{C}(j_i) \otimes_2 \cdots \otimes_2 \mathcal{C}(j_k)) \to \mathcal{C}(j)$ also exists. For instance $\gamma^{12} : \mathcal{C}(3) \otimes_1 (\mathcal{C}(1) \otimes_2 \mathcal{C}(2) \otimes_2 \mathcal{C}(3)) \to \mathcal{C}(6)$ appears as the relation



5.5**. Example.** Now we consider some products of the previous two described operads in $\mathrm{Seq}(\mathbb{N}, +)$. We expect $\mathcal{B} \otimes' \mathcal{C}$ given by $(\mathcal{B} \otimes' \mathcal{C})(j) = \mathcal{B}(j) \otimes_3 \mathcal{C}(j)$ to be a 2-fold operad and it is. It appears thus:



We demonstrate the tightness of the existence of products of operads by pointing out that $D(j) = \mathcal{B}(j) \otimes_2 \mathcal{C}(j)$ does not form an operad. We leave it to the reader to demonstrate this fact.

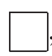
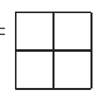
Now we pass to the categories of Young diagrams in which the interesting products are given by horizontal and vertical stacking. It is important that we do not restrict the morphisms to those between diagrams of the same total number of blocks in order that all the operad compositions exist.

5.6**. Theorem.** *A sequence of Young diagrams* $\mathcal{C}(n)$, $n \in \mathbb{N}$, *in the category* $\mathrm{ModSeq}(\mathbb{N}, +)$, *is a 2-fold operad if* $C(0) = \emptyset$ *and for* $n \geq 1$, $h(\mathcal{C}(n)) = f(n)$ *where* $f \colon \mathbb{N} \to \mathbb{N}$ *is a function such that* $f(1) = 0$ *and* $f(i + j) > f(i) + f(j)$.

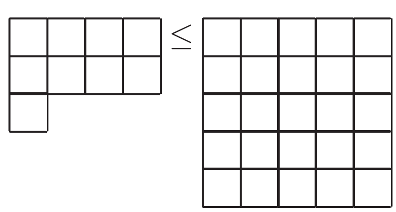
Proof. These conditions are not necessary, but they are sufficient since the first implies that $\mathcal{C}(1) = 0$ which shows that the unit conditions are satisfied; and the second implies that the maps γ exist. We see existence of γ^{12} since for $j_i > 0$, $h(\mathcal{C}(k) \otimes_1 (\mathcal{C}(j_1) \otimes_2 \cdots \otimes_2 \mathcal{C}(j_k))) = \max(f(k), \max(f(j_i))) \leq f(j)$. We have existence of γ^{13} and γ^{23} since $\max(f(k), \sum f(j_i)) \leq f(j)$. $\square$

5.7**. Example.** Examples of f include $(x-1)P(x)$ where P is a polynomial with coefficients in $\mathbb{N}$. This is easy to show since then P will be monotone increasing for $x \geq 1$ and thus $(i+j-1)P(i+j) = (i-1)P(i+j) + jP(i+j) \geq (i-1)P(i) + jP(j) - P(j)$. By this argument we can also use any $f = (x-1)g(x)$ where $g \colon \mathbb{N} \to \mathbb{N}$ is monotone increasing for $x \geq 1$.

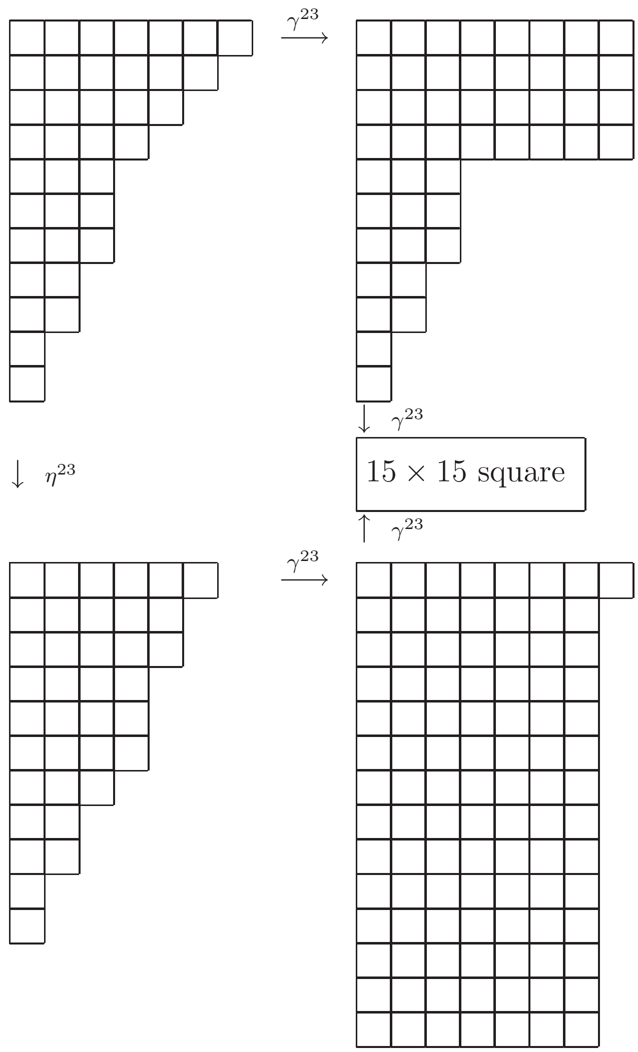
For a specific example with a handy picture that also illustrates again the nontrivial use of the interchange η we simply let $f = x - 1$. Then we have to actually describe the elements of ModSeq($\mathbb{N}$) that make up the operad. One nice choice is the operad $\mathcal{C}$ where $\mathcal{C}(n) = \{n-1, n-1, ..., n-1\}$, the $(n-1) \times (n-1)$ square Young diagram.

$\mathcal{C}(1) = 0,\ \mathcal{C}(2) =$ , $\mathcal{C}(3) =$ , $\ldots$

For instance $\gamma^{23} : \mathcal{C}(3) \otimes_2 (\mathcal{C}(1) \otimes_3 \mathcal{C}(3) \otimes_3 \mathcal{C}(2)) \to \mathcal{C}(6)$ appears as the relation



An instance of the associativity diagram with upper left position $\mathcal{C}(2) \otimes_2 (\mathcal{C}(3) \otimes_3 \mathcal{C}(2)) \otimes_2 (\mathcal{C}(2) \otimes_3 \mathcal{C}(2) \otimes_3 \mathcal{C}(4) \otimes_3 \mathcal{C}(5) \otimes_3 \mathcal{C}(3))$ is as follows:



5.8**. Example.** Again we note that the conditions in Theorem 5.6 are not neccessary ones. In fact, given any Young diagram B we can construct a unique operad that is minimal in each term with respect to ordering of the diagrams.

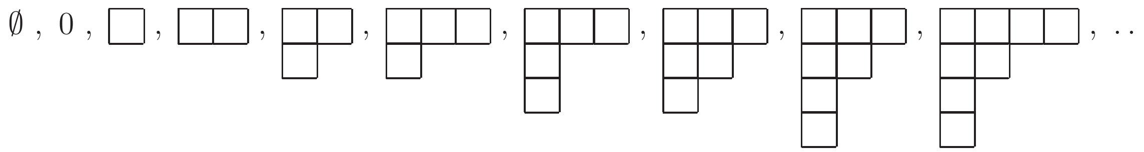
5.9**. Definition.** The 2-fold operad in the category of Young diagrams generated by a Young diagram B is denoted by $\mathcal{C}_B$ and defined as follows: $\mathcal{C}_B(1) = 0$ and $\mathcal{C}_B(2) = B$. Each sucessive term is defined to be the lexicographic maximum of all the products of prior terms which compose to the term in question; for $n > 2$ and over $\sum j_i = n$:

$$\mathcal{C}_B(n) = \max\{\mathcal{C}_B(k) \otimes_2 (\mathcal{C}_B(j_1) \otimes_3 \cdots \otimes_3 \mathcal{C}_B(j_k))\}.$$

5.10**. Theorem.** *If a Young diagram B consists of total number of blocks q, then the term $\mathcal{C}_B(n)$ of the operad generated by B consists of $q(n-1)$ blocks.*

Proof. The proof is by strong induction. The number of blocks is given for $\mathcal{C}_B(1)$ and $\mathcal{C}_B(2)$. Since the definition is in terms of a maximum over composable products, if the number of blocks in each piece of any such a product is assumed by induction to be respectively $q(k-1)$, and $q(j_1 - 1) \dots q(j_k - 1)$, then the total number of blocks in each product (and thus the maximum) is $q(n-1)$ since $\sum j_i = n$. $\square$

Here are the first few terms of the operad thus generated by $B = \square$.



Note that height of any given column grows linearly, but that the length of any row grows logarithmically.

5.11. **Theorem.** *The minimal operad $\mathcal{C}_\square$ of Young diagrams which begins with $\mathcal{C}_\square(1) = 0$ and $\mathcal{C}_\square(2) = \square$, has terms $\mathcal{C}_\square(n)$ that are built of $n-1$ blocks each, and whose monotone decreasing sequence representation is given by the formula*

$$\mathcal{C}_\square(n)_k = \text{Round}\left(n/2^k\right); k = 1, 2, \ldots$$

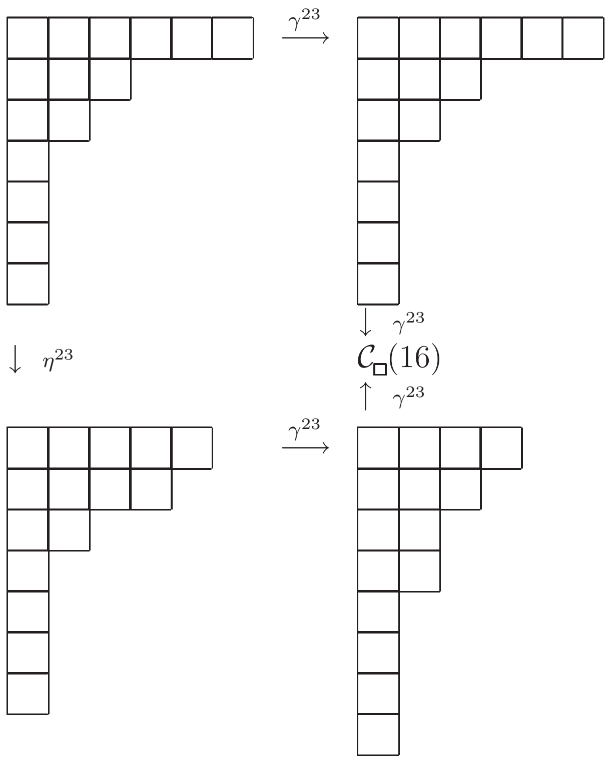
where rounding is done to the nearest integer and .5 is rounded to zero.

Proof. The proof of the formula for the column heights is by way of first showing that each term in $\mathcal{C}_\square$ can be built canonically as follows:

$$\mathcal{C}_\square(n) = \mathcal{C}_\square(\lceil \tfrac{n}{2} \rceil) \otimes_2 \left(\overbrace{\underbrace{\mathcal{C}_\square(2) \otimes_3 \cdots \otimes_3 \mathcal{C}_\square(2)}_{\lfloor \frac{n}{2} \rfloor} \otimes_3 \mathcal{C}_\square(1)}^{\lceil \frac{n}{2} \rceil} \right)$$

We must demonstrate that the maximum of all $\mathcal{C}_\square(k) \otimes_2 (\mathcal{C}_\square(j_1) \otimes_3 \cdots \otimes_3 \mathcal{C}_\square(j_k))$ where $\sum j_i = n$ is precisely given by the above canonical construction. We make the assumption (of strong induction) that this holds for terms before the n^{th} term, and check for the inequality $\mathcal{C}_\square(k) \otimes_2 (\mathcal{C}_\square(j_1) \otimes_3 \cdots \otimes_3 \mathcal{C}_\square(j_k))$ less than or equal to the canonical construction. The case in which there are only 0 or 1 odd integers among the j_k's is directly observed using the strong induction. If there are two or more odd integers among the j_k's and the first column of the diagram they help determine is greater than or equal to the first column of $\mathcal{C}_\square(k)$ then the inequality holds by induction on the size of the first column. If there are two or more odd integers among the j_k's and the first column of the diagram they help determine is less than the first column of $\mathcal{C}_\square(k)$ then we check the sub-cases n odd and n even. For n even the result is seen directly, and for n odd we again rely on induction. $\square$

For comparison to the previous example of the operad with square terms, the instance of the associativity diagram with upper left position $\mathcal{C}_\square(2) \otimes_2 (\mathcal{C}_\square(3) \otimes_3 \mathcal{C}_\square(2)) \otimes_2 (\mathcal{C}_\square(2) \otimes_3 \mathcal{C}_\square(2) \otimes_3 \mathcal{C}_\square(4) \otimes_3 \mathcal{C}_\square(5) \otimes_3 \mathcal{C}_\square(3))$ is as follows:



There may be interesting applications of the type of growth modeled by operads in iterated monoidal categories. Since the growth is in multiple dimensions it suggests applications in the science of allometric measurements, broadly used to refer to any n characteristics that grow in tandem. These measurements are often used in biological sciences to try to predict values of one characteristic from others, such as tree height from trunk diameter or crown diameter, or skeletal mass from total body mass or dimensions, or even genomic diversity from various geographical features. Allometric comparisons are often used in geology and chemistry, for instance when predicting the growth of speleothems or crystals. There are also potential applications to networks, where the growth of diameter or linking distance of a network is related logarithmically to the growth in number of nodes. In computational geometry, the number of vertices of the convex hull of n uniformly scattered points in a polygon grows as the log of n.

This sort of minimal growth in the terms of the operad could be perturbed, for example by replacing the term $\square\square$ in the above with the alternate term $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$, which would affect the later terms in turn. An interesting avenue for further investigation would be the comparison of such perturbations to determine the relative effects of a given perturbation's size and position of occurence in the sequence.

We conclude with a description of the concepts of n-fold operad algebra and of the tensor products of operad algebras.

5.12**. Definition.** Let $\mathcal{C}$ be an n-fold operad in $\mathcal{V}$. A $\mathcal{C}$-algebra is an object $A \in \mathcal{V}$ and maps

$$\theta^{pq} : \mathcal{C}(j) \otimes_p (\otimes_q^j A) \to A$$

for $n \geq q > p \geq 1$, $j \geq 0$.

(1) Associativity: The following diagram is required to commute for all $n \geq q > p \geq 1$, $k \geq 1$, $j_s \geq 0$, where $j = \sum_{s=1}^{k} j_s$.

$$\begin{tikzcd}
\mathcal{C}(k) \otimes_p (\mathcal{C}(j_1) \otimes_q \cdots \otimes_q \mathcal{C}(j_k)) \otimes_p (\otimes_q^j A) \arrow[r, "\gamma^{pq} \otimes_p \mathrm{id}"] \arrow[dd, "\mathrm{id} \otimes_p \eta^{pq}"'] & \mathcal{C}(j) \otimes_p (\otimes_q^j A) \arrow[d, "\theta^{pq}"'] \\
 & A \\
\mathcal{C}(k) \otimes_p ((\mathcal{C}(j_1) \otimes_p (\otimes_q^{j_1} A)) \otimes_q \cdots \otimes_q (\mathcal{C}(j_k) \otimes_p (\otimes_q^{j_k} A))) \arrow[r, "\mathrm{id} \otimes_p (\otimes_q^k \theta^{pq})"] & \mathcal{C}(k) \otimes_p (\otimes_q^k A) \arrow[u, "\theta^{pq}"]
\end{tikzcd}$$

(2) Units: The following diagram is required to commute for all $n \geq q > p \geq 1$.

$$\begin{tikzcd}
I \otimes_p A \arrow[r, equal] \arrow[d, "\mathcal{J} \otimes_p 1"'] & A \\
\mathcal{C}(1) \otimes_p A \arrow[ur, "\theta^{pq}"] &
\end{tikzcd}$$

5.13**. Example.** Of course the initial object is always an algebra for every operad, and every object is an algebra for the initial operad. For a slightly less trivial example we turn to the height preordered category of Remark 4.13. Define the operad $\mathcal{B}(j)$ as in Example 5.3. Then any nonzero sequence A is an algebra for this operad.

5.14. *Remark.* Let $\mathcal{C}$ and $\mathcal{D}$ be m-fold operads in an n-fold monoidal category. If A is an algebra of $\mathcal{C}$ and B is an algebra of $\mathcal{D}$ then $A \otimes_{i+m} B$ is an algebra for $\mathcal{C} \otimes_i' \mathcal{D}$.

That the product of n-fold operad algebras is itself an n-fold operad algebra is easy to verify once we note that the new θ is in terms of the two old ones:

$$\theta^{pq}_{A \otimes_{i+m} B} = (\theta^{pq}_A \otimes_{i+m} \theta^{pq}_B) \circ \eta^{p(i+m)} \circ (1 \otimes_p \eta^{q(i+m)})$$

Maps of operad algebras are straightforward to describe–they are required to preserve structure; that is to commute with θ.

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